Problem session 8

Problem 1. Show that if k is a field, then

1. $\mathrm{Cl}(\mathbf{A}_k^n) = 0$.
2. We can define a group morphism $\deg\colon \mathrm{Div}(\mathbb{P}_k^n) \to \mathbb{Z}$ as follows. If H is a closed irreducible subset of $\mathbb{P}_k^n$, then H is defined by a polynomial F, and we put $\deg(H) := \deg(F)$. If $D = \sum_i a_i H_i$ is a Weil divisor, then we put $\deg(D) := \sum_i a_i \deg(H_i)$. Show that this induces an isomorphism
$$\mathrm{Cl}(\mathbb{P}_k^n) \simeq \mathbb{Z}.$$

Problem 2. Let X be an integral Noetherian scheme and U an open subset of X. Show that if $Y_1, \ldots, Y_r$ are the irreducible components of $X \smallsetminus U$ that have codimension one in X, then there is an exact sequence

$$\mathbb{Z}^r \xrightarrow{f} \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0,$$

where $f(a_1, \ldots, a_r)$ is the class of $\sum_i a_i Y_i$.

Problem 3. Show that if A is a Noetherian domain, then A is factorial if and only if every codimension one prime ideal in A is principal. Deduce that A is factorial if and only if $X = \mathrm{Spec}(A)$ is normal and $\mathrm{Cl}(X) = 0$.

Problem 4. Let k be a field. Show that if $X = \mathrm{Spec}\, k[u, v, w]/(uv - w^2)$, then X is normal and $\mathrm{Cl}(X) \simeq \mathbb{Z}/2\mathbb{Z}$.

Problem 5. Let k be a field and H a hypersurface of degree d in $\mathbb{P}^n$ (that is, a codimension one closed subset defined by a polynomial of degree d). Show that $\mathrm{Cl}(\mathbb{P}^n \smallsetminus H) \simeq \mathbb{Z}/d\mathbb{Z}$.