Problem session 8

Problem 1. Show that if $k$ is a field, then

1. $\text{Cl}(A^n_k) = 0$.
2. We can define a group morphism $\deg: \text{Div}(\mathbb{P}^n_k) \to \mathbb{Z}$ as follows. If $H$ is a closed irreducible subset of $\mathbb{P}^n_k$, then $H$ is defined by a polynomial $F$, and we put $\deg(H) := \deg(F)$. If $D = \sum_i a_i H_i$ is a Weil divisor, then we put $\deg(D) := \sum_i a_i \deg(H_i)$. Show that this induces an isomorphism $\text{Cl}(\mathbb{P}^n_k) \simeq \mathbb{Z}$.

Problem 2. Let $X$ be an integral Noetherian scheme and $U$ an open subset of $X$. Show that if $Y_1, \ldots, Y_r$ are the irreducible components of $X \setminus U$ that have codimension one in $X$, then there is an exact sequence

$$\mathbb{Z}^r \xrightarrow{f} \text{Cl}(X) \to \text{Cl}(U) \to 0,$$

where $f(a_1, \ldots, a_r)$ is the class of $\sum a_i Y_i$.

Problem 3. Show that if $A$ is a Noetherian domain, then $A$ is factorial if and only if every codimension one prime ideal in $A$ is principal. Deduce that $A$ is factorial if and only if $X = \text{Spec}(A)$ is normal and $\text{Cl}(X) = 0$.

Problem 4. Let $k$ be a field. Show that if $X = \text{Spec} k[u, v, w]/(uv - w^2)$, then $X$ is normal and $\text{Cl}(X) \simeq \mathbb{Z}/2\mathbb{Z}$.

Problem 5. Let $k$ be a field and $H$ a hypersurface of degree $d$ in $\mathbb{P}^n$ (that is, a codimension one closed subset defined by a polynomial of degree $d$). Show that $\text{Cl}(\mathbb{P}^n \setminus H) \simeq \mathbb{Z}/d\mathbb{Z}$.