Exercises due Monday, Nov 26, 2007

1. Let R be an integral domain which is finitely generated over a field k, let $K = \operatorname{Frac} R$, let L/K be a finite extension of fields (not necessarily separable), and $S = R'_L$. Prove that S is module-finite over R.[1]

2. Let R be an integral domain and K its fraction field. Say that an element $x \in K$ is *almost integral* over R if there exists $0 \neq c \in R$ such that $cx^t \in R$ for all $t \geq 0$.

   (a) Let $x = \frac{a}{b} \in K$, where $0 \neq b, a \in R$. Show that x is integral over R iff there is some $n \geq 1$ such that for all $k \geq 0$, $a^{n+k-1} \in b^k(a,b)^{n-1}$.

   (b) Show that every element of R' is almost integral over R.

   (c) If R is a Krull domain, show that no element of $K \setminus R$ is almost integral over R

3. Let M be an R-module with a filtration (M_k) (and the corresponding filtration topology), and L a submodule of M. Demonstrate the following topological facts.

   (a) M is discrete $\iff$ $M_k = 0$ for some $k \in \mathbb{N}$.

   (b) The closure of L in M is $\bigcap_k (L + M_k)$.

   (c) L is open $\iff$ $\exists k$ such that $M_k \subseteq L$.

   (d) Every open submodule of M is closed.

   (e) Let $\psi : M \to \hat{M}$ be the natural map, and give L the induced filtration. Then the closure of $\psi(L)$ in $\hat{M}$ is $\hat{L}$, and the closure of L in M is $\psi^{-1}(\hat{L})$.

   (f) If I is an ideal and (M_k) is an I-filtration, then $\hat{M}$ is a $\hat{R}$-module.

   (g) If N is another filtered module and $f : M \to N$ is R-linear, then f is continuous $\iff$ $\forall n \exists k_n$ such that $f(L_{k_n}) \subseteq M_n$.

4. (a) If R is Noetherian, I is an ideal, $L \subseteq M$ are finitely generated R-modules, and M is topologized via the I-adic filtration (i.e. $M_k = I^k M$), then L is closed in M $\iff$ for all $\mathfrak{p} \in \operatorname{Ass}_R(M/L)$, $\mathfrak{p} + I \neq R$.

   (b) Conclude that if R is a Noetherian ring and I an ideal, then every submodule of every finitely generated R-module is I-adically closed if and only if I is contained in the Jacobson radical of R.

5. Let R be a Noetherian ring, and $\mathfrak{m}$ a maximal ideal. Show that $\widehat{R_\mathfrak{m}} \cong \widehat{R}^\mathfrak{m}$. Find a counterexample to the statement that if R is local and $\mathfrak{p}$ is prime, $(\widehat{R}^\mathfrak{p})_{\mathfrak{p}\widehat{R}^\mathfrak{p}} \cong \widehat{R_\mathfrak{p}}^{\mathfrak{p}R_\mathfrak{p}}$. (Hence, one says that "Completion does not commute with localization.")

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[1] *Hint:* In the char p case, you'll need to review your field theory. Also, if your proof is like mine, you'll need to demonstrate and use the fact that if k'/k is a finite extension, then $k'[X_1, \ldots, X_d]$ is module-finite over $k[X_1^{p^e}, \ldots, X_d^{p^e}]$, where $X_1, \ldots, X_d$ are indeterminates.