Exercises due Monday, Nov 5, 2007 – partial solutions

1. If R is a Noetherian ring, show that the following are equivalent:

   (a) For any minimal prime $\mathfrak{p}$ of R, $R/\mathfrak{p}$ is a G-domain.

   (b) $\dim R \leq 1$ and $\#(\operatorname{Spec} R) < \infty$.

   (c) $\#(\operatorname{Spec} R) < \infty$

   (d) R has only finitely many height one primes.

   Solution. (b) $\implies$ (a): First, we show that if R is a domain, it is a G-domain. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the height 1 primes of R. By the Height Theorem, there exist nonzero $x_1, \ldots, x_t \in R$, such that for each $1 \leq j \leq t$, $\mathfrak{p}_j$ is minimal over (x_j). Let $x = \prod_{j=1}^{t} x_j$. Then for any nonzero prime ideal $\mathfrak{p}$ of R, $\mathfrak{p}$ contains some $\mathfrak{p}_i$, hence it contains x_i, hence x. Thus, $\operatorname{Spec} R_x = \{(0)\}$, so R_x is a field and R is a G-domain.

   In the general case, let $\mathfrak{p}$ be a minimal prime of R. Then $\dim R/\mathfrak{p} \leq \dim R \leq 1$ and $\#(\operatorname{Spec} R/\mathfrak{p}) \leq \#(\operatorname{Spec} R) < \infty$, so $R/\mathfrak{p}$ is a G-domain.

   (b) $\implies$ (c) $\implies$ (d): Trivial.

   (d) $\implies$ (b): Since R has only finitely many minimal primes, it is enough to show that $\dim R \leq 1$, and we may assume $\dim R \neq 0$. Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ be the height one primes of R, and let $\mathfrak{m}$ be a maximal ideal of R. Then for any $0 \neq x \in \mathfrak{m}$, there is some minimal prime $\mathfrak{p}$ of (x), in which case $\operatorname{ht} \mathfrak{p} = 1$ by the PIT, and thus $\mathfrak{p} = \mathfrak{p}_i$ for some i. Hence, $\mathfrak{m} \subseteq \bigcup S$, a finite set of primes, so by Prime Avoidance there is some j with $\mathfrak{m} = \mathfrak{p}_j$.

   (a) $\implies$ (c): First assume that R is an integral domain, so that the only minimal prime is (0). Let $K = \operatorname{Frac} R$. Then for some $0 \neq x \in R$, $R_x \cong K$. Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be the minimal primes over (x). We have $\operatorname{ht} \mathfrak{p}_i = 1$ by the Principal Ideal Theorem. Moreover, since $\operatorname{Spec} R_x = \operatorname{Spec} K = \{(0)\}$, any nonzero prime ideal $\mathfrak{p}$ of R contains x, and thus it contains some $\mathfrak{p}_i$, so these are the only height 1 primes of R. Hence, (d) holds, but (d) $\implies$ (c), so (c) holds.

   In the general case, we now know that $\#(\operatorname{Spec}(R/\mathfrak{p})) = \#(V(\mathfrak{p})) < \infty$ for all minimal primes $\mathfrak{p}$. But there are only finitely many minimal primes, and $\operatorname{Spec} R = \bigcup \{V(\mathfrak{p}) \mid \mathfrak{p} \text{ a minimal prime}\}$, so $\#(\operatorname{Spec} R) < \infty$.

   □

2. Let $R \subseteq S$ be rings, $s \in S$, J an ideal of S, and W a multiplicative subset of R.

   (a) Show that $R'_S/(J \cap R'_S) \cong (R/(J \cap R))'_{S/J}$.

   (b) Show that $(W^{-1}R)'_{W^{-1}S} \cong W^{-1}(R'_S)$.

(c) Show that $s \in R'_S$ iff $\frac{s}{1} \in (R_{\mathfrak{m}})'_{S_{\mathfrak{m}}}$[1] for all maximal ideals $\mathfrak{m}$ of R.

Solution. (a) First, note that the terms make sense: If J is an ideal of S, then there are injective ring homomorphisms $\alpha : R/(J \cap R) \hookrightarrow R'_S/(J \cap R'_S)$ and $\beta : R'_S/(J \cap R'_S) \hookrightarrow S/J$, given by $\alpha(x + (J \cap R)) = x + (J \cap R'_S)$ and $\beta(y + (J \cap R'_S)) = y + J$. We will show that if these are identified as inclusions, then we actually get equality.

Indeed, let $x + (J \cap R'_S) \in R'_S/(J \cap R'_S)$, so that $x \in R'_S$. Then $x^n + a_1 x^{n-1} + \cdots + a_n = 0$, where the $a_j \in R$. Then in S/J, we have $\overline{x}^n + \overline{a_1 x}^{n-1} + \cdots + \overline{a_n} = \overline{0}$, and each $\overline{a_j} \in R/(J \cap R)$, so $x + J \in (R/(J \cap R))'_{S/J}$.

Conversely, suppose $x + J \in (R/(J \cap R))'_{S/J}$. Then for some $a_1, \dots, a_n \in R$ and $j \in J \cap R$, we have $x^n + a_1 x^{n-1} + \cdots + a_n - j = 0$, so that $x \in R'_S$ and $x + (J \cap R'_S) \in R'_S/(J \cap R'_S)$.

(b) Both $W^{-1}R$ and $W^{-1}(R'_S)$ can be considered to be subrings of $W^{-1}S$. With this identification, we will show equality.

Indeed, let $\frac{x}{w} \in W^{-1}S$ such that $\frac{x}{w} \in W^{-1}R'_S$. Then $\frac{x}{w} = \frac{y}{v}$, where $y \in R'_S$ and $v \in W$. Then we have $y^n + a_1 y^{n-1} + \cdots + a_n = 0$, where $a_j \in R$. Thus,

$$0 = \frac{y^n + a_1 y^{n-1} + \cdots + a_n}{v^n} = \left(\frac{y}{v}\right)^n + \frac{a_1}{v}\left(\frac{y}{v}\right)^{n-1} + \cdots + \frac{a_n}{v^n},$$

and each $\frac{a_j}{v^j} \in W^{-1}R$, so $\frac{x}{w} = \frac{y}{v} \in (W^{-1}R)'_{W^{-1}S}$.

Conversely, let $\frac{x}{w} \in (W^{-1}R)'_{W^{-1}S}$. Then there exist $v_1, \dots, v_n \in W$ and $a_1, \dots, a_n \in R$ such that

$$\left(\frac{x}{w}\right)^n + \sum_{j=1}^{n} \frac{a_j}{v_j}\left(\frac{x}{w}\right)^{n-j} = 0.$$

Let $v = \prod_{j=1}^{n} v_j$ (still an element of W) and $u_j = \prod_{i \neq j} v_i$ (so that $u_j v_j = v$). Then multiplying the displayed equation by $\frac{v^{n-1}}{1}$, we get

$$\begin{aligned} 0 &= \frac{v^{n-1}x^n}{w^n} + \sum_{j=1}^{n} \frac{a_j v^{n-1} x^{n-j}}{v_j w^{n-j}} = \frac{v^n x^n}{v w^n} + \sum_{j=1}^{n} \frac{a_j u_j w^j v^{n-1} x^{n-j}}{v w^n} \\ &= \frac{(vx)^n + \sum_{j=1}^{n} a_j u_j w^j v^{j-1} (vx)^{n-j}}{v w^n}. \end{aligned}$$

Let $b_j := a_j u_j w^j v^{j-1}$ for $1 \leq j \leq n$. Then for some $t \in W$, we have that t^n times the numerator of the last displayed fraction above is zero. That is:

$$0 = t^n \left((vx)^n + \sum_{j=1}^{n} b_j (vx)^{n-j} \right) = (tvx)^n + \sum_{j=1}^{n} b_j t^j (tvx)^{n-j}.$$

[1] Here $S_{\mathfrak{m}} := (R \setminus \mathfrak{m})^{-1} S$.

Hence, $tvx \in R'_S$. But $tv \in W$, so $\frac{x}{w} \in W^{-1}(R'_S)$.

(c) Suppose that $\frac{s}{1} \in (R_\mathfrak{m})'_{S_\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R. Then by part (b), $\frac{s}{1} \in (R'_S)_\mathfrak{m}$ for all $\mathfrak{m}$. But R'_S is an R-submodule of S and membership in a submodule is a local property, so we have $s \in R'_S$. Conversely, if $s \in R'_S$, then by part (b), letting $W = R \setminus \mathfrak{m}$ we have $\frac{s}{1} \in W^{-1}(R'_S) = (W^{-1}R)'_{W^{-1}S} = (R_\mathfrak{m})'_{S_\mathfrak{m}}$.

□

3. Let $\varphi : R \to S$ be a homomorphism of Noetherian rings, $\mathfrak{q} \in \operatorname{Spec} S$, $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, M a finitely generated R-module, and N a finitely-generated S-module. Show that $(M \otimes_R N)_\mathfrak{q} \cong M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{q}$ as right $S_\mathfrak{q}$-modules.

Solution. Actually we don't need the fact that R is Noetherian! All we need is that M is *finitely presented*. Accordingly, take a finite presentation

$$R^t \xrightarrow{(a_{ij})} R^s \to M \to 0$$

of M, where (a_{ij}) is an $s \times t$ matrix of elements of R, which give the action of the map from a fixed basis of R^t to a fixed basis of R^s. If we localize at $\mathfrak{p}$, we get the exact sequence

$$R^t_\mathfrak{p} \xrightarrow{\left(\frac{a_{ij}}{1}\right)} R^s_\mathfrak{p} \to M_\mathfrak{p} \to 0.$$

Next, tensor the sequence with $N_\mathfrak{q}$ over $R_\mathfrak{p}$, to get the following commutative diagram with exact row:

$$\begin{array}{ccccccc} R^t_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{q} & \xrightarrow{\left(\frac{a_{ij}}{1} \otimes 1\right)} & R^s_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{q} & \longrightarrow & M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{q} & \longrightarrow & 0 \\ \| & & \| & & & & \\ N^t_\mathfrak{q} & \xrightarrow{g = \left(\frac{\varphi(a_{ij})}{1}\right)} & N^s_\mathfrak{q} & & & & \end{array}$$

Hence, $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{q} \cong \operatorname{Cok} g$. On the other hand, if we tensor the original presentation of M with N over R, we get the commutative diagram with exact row:

$$\begin{array}{ccccccc} R^t \otimes_R N & \xrightarrow{(a_{ij} \otimes 1)} & R^s \otimes_R N & \longrightarrow & M \otimes_R N & \longrightarrow & 0 \\ \| & & \| & & & & \\ N^t & \xrightarrow{(\varphi(a_{ij}))} & N^s & & & & \end{array}$$

Then localizing at $\mathfrak{q}$, we get the following exact sequence (since $(N_\mathfrak{q})^j \cong (N^j)_\mathfrak{q}$ in a natural way for any j):

$$N^t_\mathfrak{q} \xrightarrow{g = \left(\frac{\varphi(a_{ij})}{1}\right)} N^s_\mathfrak{q} \to (M \otimes_R N)_\mathfrak{q} \to 0,$$

with the same g we had earlier! Thus, $(M \otimes_R N)_{\mathfrak{q}} \cong \operatorname{Cok} g \cong M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{q}}$. $\square$

4. (a) Let R be a Noetherian ring. Show that R is catenary if and only if for every minimal prime $\mathfrak{p}$ of R and every $\mathfrak{q}, \mathfrak{q}' \in \operatorname{Spec} S$ such that $\mathfrak{q} \subset \mathfrak{q}'$ is a saturated chain of primes, where $S = R/\mathfrak{p}$, we have $\operatorname{ht} \mathfrak{q}' = \operatorname{ht} \mathfrak{q} + 1$.

   (b) In light of the ring $R = k[X, Y_1, \ldots, Y_n]/(XY_1, \ldots, XY_n)$ (where $n \geq 2$, k is a field, and the variables X and Y_i are indeterminates over k), explain why it was necessary to pass to S in part (a).

Solution. (a) If R is catenary and $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{q}'$ are as above, then note that S is also catenary, since the saturated chains of primes in S can be embedded as a subset of the saturated chains of primes in R. Let $h = \operatorname{ht} \mathfrak{q}'$. Then there is a saturated chain of primes

$$0 = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_{n-1} = \mathfrak{q} \subset \mathfrak{q}_n = \mathfrak{q}'.$$

Since every saturated chain of primes linking 0 to $\mathfrak{q}'$ has length h, we have $h = n$. By removing $\mathfrak{q}'$ from the above chain, we see a saturated chain of primes of length $h - 1$ linking 0 to $\mathfrak{q}$. Since S is catenary, every such saturated chain must have length $h - 1$, so $\operatorname{ht} \mathfrak{q} = h - 1 = \operatorname{ht} \mathfrak{q}' - 1$.

Conversely, supposing the given condition holds, we must show that R is catenary. Let $\mathfrak{p} \subset \mathfrak{q}$ be prime ideals of R, and let $(*) : \mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_h = \mathfrak{q}$ and $(**) : \mathfrak{p} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n = \mathfrak{q}$ be two saturated chains of primes linking $\mathfrak{p}$ to $\mathfrak{q}$. Let P be any minimal prime of R such that $P \subseteq \mathfrak{p}$. Note that $\mathfrak{q}_i/P \subset \mathfrak{q}_{i+1}/P$ is a saturated chain of primes in R/P for $0 \leq i < n$, as is $\mathfrak{p}_j/P \subset \mathfrak{p}_{j+1}/P$ for $0 \leq j < h$, by the order-preserving bijective correspondence between $V(P)$ and $\operatorname{Spec} R/P$. So $(*)/P$ and $(**)/P$ are saturated chains in R/P, whence given condition implies that $\operatorname{ht} \mathfrak{q}_{i+1}/P = 1 + \operatorname{ht} \mathfrak{q}_i/P$ for $0 \leq i < n$, and similarly for the $\mathfrak{p}_j$s. Then by induction, we have

$$\begin{aligned} n + \operatorname{ht} \mathfrak{p}/P = n + \operatorname{ht} \mathfrak{q}_0/P &= \operatorname{ht} \mathfrak{q}_n/P = \operatorname{ht} \mathfrak{q}/P \\ &= \operatorname{ht} \mathfrak{p}_h/P = h + \operatorname{ht} \mathfrak{p}_0/P = h + \operatorname{ht} \mathfrak{p}/P. \end{aligned}$$

Thus, $n = h$.

(b) First note that R is catenary, as we showed in class, since it is a finitely generated algebra over a field.

Let $x, y_1, \ldots, y_n$ be the images of $X, Y_1, \ldots, Y_n$ respectively in R. Set $\mathfrak{p}_1 := (x)$ and $\mathfrak{p}_2 := (y_1, \ldots, y_n)$. Then $R/\mathfrak{p}_1 \cong k[Y_1, \ldots, Y_n]$ and $R/\mathfrak{p}_2 \cong k[X]$ are integral domains, so each $\mathfrak{p}_i$ is prime. Moreover, if $\mathfrak{p}$ is any prime ideal of R such that $\mathfrak{p}_1 \nsubseteq \mathfrak{p}$, then $x \notin \mathfrak{p}$, so $xy_i = 0 \in \mathfrak{p}$ for all i implies that $\mathfrak{p}_2 \subseteq \mathfrak{p}$. Thus, $\operatorname{Min} R = \{\mathfrak{p}_1, \mathfrak{p}_2\}$.

Now, let $\mathfrak{q}' := (x, y_1, \dots, y_n)$ and $\mathfrak{q} := \mathfrak{p}_2$. Then $\mathfrak{q}$ is a minimal prime, so $\operatorname{ht}\mathfrak{q} = 0$. Moreover, $\mathfrak{q} \subset \mathfrak{q}'$ is a saturated chain of primes, since in the identification of $R/\mathfrak{q}$ with $k[X]$, $\mathfrak{q}'/\mathfrak{q}$ gets identified with (X), which is a height 1 prime. On the other hand, the following is a proper chain of primes in R:

$$(x) \subset (x, y_1) \subset \cdots \subset (x, y_1, \dots, y_n),$$

so $\operatorname{ht}\mathfrak{q}' \geq n > 1 = 0 + 1 = \operatorname{ht}\mathfrak{q} + 1$. Thus, we can't modify part (a) to talk about length 1 saturated chains of primes in a catenary ring R; we have to go modulo each minimal prime.

□

5. Let R be a Noetherian domain and let $0 \neq x \in R$ such that the ideal (x) is prime. Show: R is a UFD iff R_x is a UFD.

Solution. Throughout, we use the characterization that among Noetherian domains A, A is a UFD iff all height 1 primes of A are principal.

Suppose R is a UFD. Let $\mathfrak{p} = \mathfrak{q}R_x$ be a height one prime in R_x. Let $\mathfrak{q}_0 \in \operatorname{Spec} R$ with $\mathfrak{q}_0 \subsetneq \mathfrak{q}$. Then since the localization map $\ell : R \to R_x$ is injective (as x is a non-zerodivisor), we have $\mathfrak{q}_0 R_x \subsetneq \mathfrak{q}R_x$, and hence $\mathfrak{q}_0 R_x = 0$. Again by injectivity of ℓ, we have $\mathfrak{q}_0 = 0$. So $\operatorname{ht}\mathfrak{q} = 1$. Since R is a UFD, $\mathfrak{q}$ is principal, which implies that $\mathfrak{p} = \mathfrak{q}R_x$ is principal. Thus R_x is a UFD.

Conversely, suppose R_x is a UFD, and let $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{ht}\mathfrak{p} = 1$. If $x \in \mathfrak{p}$, then since (x) is itself a height one prime, we have $(x) = \mathfrak{p}$. If $x \notin \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}R_x$ is a height one prime in R_x, so $\mathfrak{p}R_x = \left(\frac{a}{1}\right)$ for some $0 \neq a \in \mathfrak{p}$.

Since each (x^n) is (x)-primary, the Krull intersection theorem shows that

$$\bigcap_{n \in \mathbb{N}} (x^n) = \bigcap_n \left((x^n)R_{(x)} \cap R\right) = \left(\bigcap_n (x^n)R_{(x)}\right) \cap R = 0 \cap R = 0.$$

Thus, there is some n such that $a \in (x^n) \setminus (x^{n+1})$. Then $a = x^n b$ for some $b \in R$, and $b \notin (x)$. Moreover, since $\frac{x^n}{1}$ is a unit in R_x, we have $\mathfrak{q} = \left(\frac{a}{1}\right) = \left(\frac{x^n}{1} \cdot \frac{b}{1}\right) = \left(\frac{b}{1}\right)$.

Now, if $y \in \mathfrak{p}$, then $\frac{y}{1} \in \mathfrak{p}R_x = bR_x$, so $\frac{y}{1} = \frac{bc}{x^t}$ for some $c \in R$, $t \in \mathbb{N}$. Thus, $x^{t+j}y = x^j bc$ for some j, so $bc = x^t y \in (x^t)$, but $b \notin (x)$ and (x^t) is (x)-primary, so $c \in (x^t)$. That is, $c = x^t e$ for some $e \in R$, so $bx^t e = x^t y$, whence $y = be \in (b)$. Thus, $\mathfrak{p} \subseteq (b)$. However, since $a = x^n b \in \mathfrak{p}$ and $x \notin \mathfrak{p}$, we have $b \in \mathfrak{p}$. Thus, $\mathfrak{p} = (b)$ is principal. Thus, R is a UFD. □

6. Prove the lemma that appears immediately before the Noether Normalization theorem. (Hint: Think in base N.)

Solution. If $n = 1$, then there is nothing to prove. Accordingly, we assume that $n > 1$.

f is a sum of terms of the form $m_a = c_a X_1^{a_1} \cdots X_n^{a_n}$, where $a = (a_1, \dots, a_n)$, $0 \neq c_a \in k$ and $a_j < N$ for each j. Then $\varphi(m_a) = c_a X_n^{a_n} \prod_{i=1}^{n-1}(X_i + X_n^{N^i})^{a_i}$. Expanding out, we have that $\varphi(m_a)$ is a sum of terms of the form

$$\begin{aligned} t_j &:= n_j c_a X_n^{a_n} \prod_{i=1}^{n-1} X_i^{j_i} X_n^{N^i(a_i - j_i)} \\ &= n_j c_a X_n^{a_n + \sum_{i=1}^{n-1} N^i(a_i - j_i)} \prod_{i=1}^{n-1} X_i^{j_i}, \end{aligned}$$

where $j = (j_1, \dots, j_{n-1})$, $0 \leq j_i \leq a_i$, $n_j \in \mathbb{N}$, and $n_{(0,\dots,0)} = 1$.

We want to find the nonzero term(s) in $\varphi(m_a)$ with the greatest X_n-degree. Since $t_0 := t_{(0,\dots,0)} \neq 0$, it stands as a candidate. Pick any nonzero term t_j with $j \neq (0, \dots, 0)$. Without loss of generality, $h = 1$. Then

$$\deg_{X_n} t_0 = a_n + \sum_{i=1}^{n-1} N^i a_i > a_n + \sum_{i=1}^{n-1} N^i(a_i - j_i) = \deg_{X_n} t_j.$$

Hence $t_0 = c_a X_n^{a_n + \sum_{i=1}^{n-1} N^i a_i}$ is the *unique* nonzero term of $\varphi(m_a)$ of highest X_n-degree.

Now, if $m_a = c_a X_1^{a_1} \cdots X_n^{a_n}$ and $m_b = c_b X_1^{b_1} \cdots X_n^{b_n}$ are two distinct terms of f, then the highest X_n-degree terms of $\varphi(m_a)$ and $\varphi(m_b)$ are $c_a X_n^{a_n + \sum_{i=1}^{n-1} N^i a_i}$ and $c_b X_n^{b_n + \sum_{i=1}^{n-1} N^i b_i}$ respectively. I claim that one of these has strictly higher X_n-degree than the other. For if $a_n + \sum_{i=1}^{n-1} N^i a_i = b_n + \sum_{i=1}^{n-1} N^i b_i$, then by unique representation of integers in base N (since $a_j < N$ and $b_j < N$ for each j), we have $a_i = b_i$ for each i, and hence $a = b$, contradicting the distinctness of the terms.

Thus, $\varphi(f)$ has a unique highest X_n-degree term, and this term is the t_0 of some a, which has the required form. □

7. (The Dimension Inequality / Formula): Let $R \subseteq S$ be Noetherian integral domains, such that S is a finitely-generated R-algebra. Let K, L be the fraction fields of R, S, respectively. Let $\mathfrak{q} \in \operatorname{Spec} S$ and $\mathfrak{p} = \mathfrak{q} \cap R$. Then

$$\operatorname{ht} \mathfrak{q} + \operatorname{tr.deg.}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \leq \operatorname{ht} \mathfrak{p} + \operatorname{tr.deg.}_K L,$$

with equality if R is universally catenary.