Exercises due Friday, Nov 9, 2007

1. If R is a Noetherian ring, show that the following are equivalent:

   (a) For any minimal prime $\mathfrak{p}$ of R, $R/\mathfrak{p}$ is a G-domain.

   (b) $\dim R \leq 1$ and $\#(\operatorname{Spec} R) < \infty$.

   (c) $\#(\operatorname{Spec} R) < \infty$

   (d) R has only finitely many height one primes.

2. Let $R \subseteq S$ be rings, $s \in S$, J an ideal of S, and W a multiplicative subset of R.

   (a) Show that $R'_S/(J \cap R'_S) \cong (R/(J \cap R))'_{S/J}$.

   (b) Show that $(W^{-1}R)'_{W^{-1}S} \cong W^{-1}(R'_S)$.[1]

   (c) Show that $s \in R'_S$ iff $\frac{s}{1} \in (R_\mathfrak{m})'_{S_\mathfrak{m}}$ [2] for all maximal ideals $\mathfrak{m}$ of R.

3. Let $\varphi : R \to S$ be a homomorphism of Noetherian rings, $\mathfrak{q} \in \operatorname{Spec} S$, $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, M a finitely generated R-module, and N an S-module. Show that $(M \otimes_R N)_\mathfrak{q} \cong M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{q}$ as right $S_\mathfrak{q}$-modules.[3]

4. (a) Let R be a Noetherian ring. Show that R is catenary if and only if for every minimal prime $\mathfrak{p}$ of R and every $\mathfrak{q}, \mathfrak{q}' \in \operatorname{Spec} S$ such that $\mathfrak{q} \subset \mathfrak{q}'$ is a saturated chain of primes, where $S = R/\mathfrak{p}$, we have $\operatorname{ht} \mathfrak{q}' = \operatorname{ht} \mathfrak{q} + 1$.

   (b) In light of the ring $R = k[X, Y_1, \ldots, Y_n]/(XY_1, \ldots, XY_n)$ (where $n \geq 2$, k is a field, and the variables X and Y_i are indeterminates over k), explain why it was necessary to pass to S in part (a).

5. Let R be a Noetherian domain and let $0 \neq x \in R$ such that the ideal (x) is prime. Show: R is a UFD iff R_x is a UFD.

6. Prove the Preparation Lemma that appears before the Noether Normalization theorem. (Hint: Think in base N.)

7. [4] (The Dimension Inequality / Formula): Let $R \subseteq S$ be Noetherian integral domains, such that S is a finitely-generated R-algebra. Let K, L be the fraction fields of R, S, respectively. Let $\mathfrak{q} \in \operatorname{Spec} S$ and $\mathfrak{p} = \mathfrak{q} \cap R$. Then

$$\operatorname{ht} \mathfrak{q} + \operatorname{tr.deg.}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \leq \operatorname{ht} \mathfrak{p} + \operatorname{tr.deg.}_K L,$$

   with equality if R is universally catenary.

[1] This one takes a lot of bookkeeping. Be careful.
[2] Here $S_\mathfrak{m} := (R \setminus \mathfrak{m})^{-1}S$.
[3] This provides the missing step from the proof on Oct 19 that $\dim R[X] = \dim R + 1$.
[4] Not required. Extra credit!