Exercises due Wednesday, Oct 24, 2007 – solutions

1. (a) Let $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a right-exact sequence of R-modules. Show that if L, N are finitely generated, so is M, and that if M is finitely generated, so is N.

   (b) Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence of R-modules such that M is finitely generated and N is finitely presented. Show that L is finitely generated.

   (c) Conclude that for any finitely presented R-module N and any surjection $\pi : R^s \twoheadrightarrow N$, $\operatorname{Ker}\pi$ is finitely generated.

Solution. (a) First suppose L and N are finitely generated. Let $z_1, \ldots, z_n$ be a generating set for N. Let $y_1, \ldots, y_n \in M$ such that $g(y_i) = z_i$. Let $x_1, \ldots, x_t$ be a generating set for L. Then I claim that M is generated by the set $\{f(x_1), \ldots, f(x_t), y_1, \ldots, y_n\}$. For let $y \in M$. Then there exist $a_1, \ldots, a_n \in R$ with $g(y) = \sum_{i=1}^n a_i z_i = g(\sum_{i=1}^n a_i y_i)$, so that $y - \sum_{i=1}^n a_i y_i \in \operatorname{Ker} g = \operatorname{Im} f$, so that there exist $b_1, \ldots, b_t \in R$ with $y - \sum_{i=1}^n a_i y_i = f(\sum_{j=1}^t b_j x_j) = \sum_{j=1}^t b_j f(x_j)$.

Next, suppose M is finitely generated. Let $y_1, \ldots, y_n$ be a generating set. Then the elements $g(y_1), \ldots, g(y_n)$ generate N because g is surjective.

(b) There are two ways I know of to do this problem. One of them can be found in the first chapter of Matsumura's book: ”Commutative Ring Theory”. Here's another:

We know that there exist $t, n \in \mathbb{N}$ and a finitely generated module K such that there is a short exact sequence $0 \to K \to R^n \xrightarrow{\gamma} N \to 0$ and a surjection $\beta : R^t \twoheadrightarrow L$. Since R^n is projective and g surjective, there exists $\widetilde{\gamma} : R^n \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
 & R^n \arrow[dl, "\widetilde{\gamma}"'] \arrow[d, two heads, "\gamma"] \\
M \arrow[r, two heads, "g"] & N
\end{tikzcd}$$

Since R^t is projective and $g \circ \beta$ surjective, there is a map $\widetilde{g} : R^t \to R^n$ that makes the following diagram commute:

$$\begin{tikzcd}
R^t \arrow[r, "\widetilde{g}"] \arrow[d, two heads, "\beta"] & R^n \arrow[d, two heads, "\gamma"] \\
M \arrow[r, two heads, "g"] & N
\end{tikzcd}$$

Define $g' : R^t \oplus R^n \to R^n$ by $g'(x, y) = \widetilde{g}(x) + y$, and $\beta' : R^t \oplus R^n \to M$ by $\beta'(x, y) = \beta(x) + \widetilde{\gamma}(y)$. Then it is straightforward to check that we have the following commutative diagram of surjections:

$$\begin{tikzcd}
R^t \oplus R^n \arrow[r, two heads, "g'"] \arrow[d, two heads, "\beta'"] & R^n \arrow[d, two heads, "\gamma"] \\
M \arrow[r, two heads, "g"] & N
\end{tikzcd}$$

We can complete the top row of this diagram to a short exact sequence: $0 \to R^t \xrightarrow{f'} R^t \oplus R^n \xrightarrow{g'} R^n \to 0$, where f' is defined by $f'(x) := (x, -\widetilde{g}(x))$. Hence, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & R^t \arrow[r, "f'"] & R^t \oplus R^n \arrow[r, "g'"] \arrow[d, two heads, "\beta'"] & R^n \arrow[r] \arrow[d, two heads, "\gamma"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}$$

Moreover, $g \circ \beta' \circ f' = \gamma \circ g' \circ f' = \gamma \circ 0 = 0$, so $\beta' \circ f'$ factors through f, since f is injective and $\operatorname{Im} f = \operatorname{Ker} g$. That is, there is a map $\alpha : R^t \to L$ that gives us the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & K \arrow[d] & \\
0 \arrow[r] & R^t \arrow[r, "f'"] \arrow[d, "\alpha"] & R^t \oplus R^n \arrow[r, "g'"] \arrow[d, "\beta'"] & R^n \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & C \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Here, C is the cokernel of α.

By the Snake Lemma, we have an exact sequence $K \to C \to 0$, so that since K is finitely generated, C is finitely generated by part (a). Then since $R^t \xrightarrow{\alpha} L \to C \to 0$ is exact and both R^t and C are finitely generated, L is finitely generated by part (a).

(c) We have a short exact sequence $0 \to \operatorname{Ker} \pi \to R^s \xrightarrow{\pi} N \to 0$. Since N is finitely presented and R^s is finitely generated, by part (b) we have that $\operatorname{Ker} \pi$ is finitely generated.

$\square$

2. Show that if R is a Noetherian ring with an embedded prime, then R is not reduced. Is the converse true? (Prove or give a counterexample.)

   Solution. Suppose R has an embedded prime. Then there is a proper containment $\mathfrak{p} \subset \mathfrak{q}$ of associated primes of R. Then $\mathfrak{p} = (0 : x)$ and $\mathfrak{q} = (0 : y)$ for some $x, y \in R$. Since $\mathfrak{p}$ and $\mathfrak{q}$ are proper, $y \neq 0$. Let $a \in \mathfrak{q} \setminus \mathfrak{p}$. Then $ay = 0 \in \mathfrak{p}$, but $a \notin \mathfrak{p}$, so $y \in \mathfrak{p} \subseteq \mathfrak{q} = (0 : y)$, which means that $y^2 = 0$. Thus, y is a nonzero nilpotent element of R, so R is not reduced.

   However, this is not a necessary condition for R being reduced. For example, let $R = \mathbb{Q}[X]/(X^2)$, where X is an indeterminate. Let x be the image of X in R. Then $x^2 = 0$ and $x \neq 0$, so R is not reduced. However, $\operatorname{Spec} R = \{(x)\}$. To see this, note that for any prime ideal $\mathfrak{p}$ of R, we have $x^2 = 0 \in \mathfrak{p}$ so that $x \in \mathfrak{p}$, and hence $\mathfrak{p}$ contains (x). But (x) is a maximal ideal because $R/(x) \cong \mathbb{Q}$ is a field. Since $\operatorname{Spec} R$ only has one prime ideal, it cannot have a proper containment of prime ideals at all, much less associated prime ideals.

   The same argument works for any Artinian local ring that is not a field. $\square$

3. Let M be a k-module, where k is a field. Without using Hopkins' theorem (or anything that depends on it), show that$M$ is Artinian if and only if M is a finite-dimensional vector space over k.

   Solution. First suppose M is a finite-dimensional vector space over k. Let n be the dimension of M as a k-vector space. If $n = 0$ then $M = 0$ is obviously Artinian. So say $n > 0$ and assume inductively that any k-vector space with dimension less than n is Artinian. Let $N_0 \supset N_1 \supset \cdots$ be a descending chain of subspaces of M. Then $\dim N_1 < \dim N_0 \leq n$, so by induction, N_1 is Artinian, so the chain $N_1 \supset \cdots$ must be finite. Hence, the original chain, with just one additional link, must also be finite. Hence M is Artinian.

   On the other hand, suppose M is infinite-dimensional as a k-vector space. Then there is an infinite k-basis of M. Let $S := \{e_n \mid n \in \mathbb{N}\}$ be a linearly independent subset of M (Pick S to be part of an infinite basis of M.). For $n \in \mathbb{N}$, let L_n be the subspace of M that is spanned by $S_n := \{e_k \mid k \geq n\}$. Then $L_n \supseteq L_{n+1}$ for each n, since $S_n \supseteq S_{n+1}$, and $L_n \neq L_{n+1}$, since $e_n \notin L_{n+1}$ (by the assumption of linear independence). Thus, $L_1 \supset L_2 \supset \cdots$ forms an strict infinite descending chain of k-submodules of M, which means that M is not Artinian. $\square$

4. Prove that in Prop 1, Prop 2, and the final Corollary from the Oct 8 lecture notes, the conclusions hold if all the assumptions on R and M are replaced with the single assumption that M is a Noetherian module.[1]

   Solution. Let R be a ring, $W \subseteq R$ a multiplicative set, and M a Noetherian R-module.

   Let $\bar{R}$ be as in the hint, and $I := \operatorname{Ann} M$. Then we showed in class that $\bar{R}$ is a Noetherian ring, and M is obviously Noetherian as an $\bar{R}$-module. I claim there is an inclusion-preserving bijective correspondence $g : \operatorname{Ass}_R M \to \operatorname{Ass}_{\bar{R}} M$, given by $\mathfrak{p} \mapsto \bar{\mathfrak{p}}$. Inclusion-preservation is clear, assuming well-definedness.

   Suppose $\mathfrak{p} \in \operatorname{Ass}_R M$. Then $\mathfrak{p} = (0 : z)$ for some $z \in M$. Then every element of $\bar{\mathfrak{p}}$ annihilates z, and for any $\bar{a} \in \bar{R}$ such that $\bar{a}z = 0$, this means that there is some $i \in I$ with $0 = (a+i)z = az + iz = az$, so that $a \in \mathfrak{p}$ and $\bar{a} \in \bar{\mathfrak{p}}$. Hence $\bar{\mathfrak{p}} = \operatorname{Ann}_{\bar{R}} z$, so $\bar{\mathfrak{p}} \in \operatorname{Ass}_{\bar{R}} M$. Thus, g is well-defined.

   To see that g is injective, suppose $\mathfrak{p}, \mathfrak{q} \in \operatorname{Ass}_R M$ with $\bar{\mathfrak{p}} = \bar{\mathfrak{q}}$. Say $\mathfrak{p} = (0 : z)$ and $\mathfrak{q} = (0 : y)$. Then since I annihilates both y and z, we have $I \subseteq \mathfrak{p} \cap \mathfrak{q}$, so $\mathfrak{p} = \mathfrak{p} + I = \mathfrak{q} + I = \mathfrak{q}$.

   Finally we check that g is surjective. Let $\mathfrak{p} \in \operatorname{Spec} R$ with $\bar{\mathfrak{p}} = \operatorname{Ann}_{\bar{R}} z$ for some $z \in M$. Then for any $a \in \mathfrak{p}$, there is some $i \in I$ with $0 = (a+i)z = az$, so $a \in (0 : z)$. Conversely, if $a \in R$ with $az = 0$, then $\bar{a}z = 0$ so $\bar{a} \in \bar{\mathfrak{p}}$, and hence $a + i \in \mathfrak{p}$ for some $i \in I$, but $i \in \mathfrak{p}$, so $a \in \mathfrak{p}$. Thus, $\mathfrak{p} = \operatorname{Ann}_R z \in \operatorname{Ass}_R M$, and $g(\mathfrak{p}) = \bar{\mathfrak{p}}$.

   The correspondence above shows not only that g is bijective, but that for any given z, $\mathfrak{p}$ is the annihilator of z in R if and only if $\bar{\mathfrak{p}}$ is the annihilator of z in $\bar{R}$.

   Moreover, for any multiplicative set W, $W^{-1}M \cong \bar{W}^{-1}M$, where $\bar{W}$ is the image of W in $\bar{R}$. Hence, g extends to an inclusion-preserving bijective correspondence (also called g) between $\operatorname{Supp}_R M$ and $\operatorname{Supp}_{\bar{R}} M$, given by $\mathfrak{p} \mapsto \bar{\mathfrak{p}}$.

   (1a): We need to show that $M = 0 \iff \operatorname{Ass} M = \emptyset$. If $M = 0$ then $\operatorname{Supp} M = \operatorname{Ass} M = \emptyset$. So suppose $M \neq 0$. Then $\operatorname{Ass}_{\bar{R}} M \neq \emptyset$ by the Proposition, so by the correspondence above, $\operatorname{Ass}_R M \neq \emptyset$.

   (1b): We need to show that $\operatorname{Ass}_{W^{-1}R} W^{-1}M = \{W^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_R M, \mathfrak{p} \cap W = \emptyset\}$. We have $\mathfrak{q} \in \operatorname{Ass}_{W^{-1}R} W^{-1}M$ iff $\bar{\mathfrak{q}} \in \operatorname{Ass}_{\bar{W}^{-1}\bar{R}} \bar{W}^{-1}M$ (since it is clear that $W^{-1}M = \bar{W}^{-1}M$) iff $\bar{\mathfrak{q}} = \bar{W}^{-1}\bar{\mathfrak{p}}$ for some $\bar{\mathfrak{p}} \in \operatorname{Ass}_{\bar{R}} M$ with $\bar{\mathfrak{p}} \cap \bar{W} = \emptyset$ (by the Proposition) iff $\mathfrak{q} = W^{-1}\mathfrak{p}$, where $\mathfrak{p} \in \operatorname{Ass}_R M$ and $\mathfrak{p} \cap W = \emptyset$ (by the correspondence).

   (1c): We need to show the minimal elements of $\operatorname{Supp}_R M$ are in $\operatorname{Ass}_R M$. By the Proposition the minimal elements of $\operatorname{Supp}_{\bar{R}} M$ are in $\operatorname{Ass}_{\bar{R}} M$, and then use g.

---

[1] *Hint:* Let $\bar{R} := R/\operatorname{Ann} M$; M is also an $\bar{R}$-module. Start by establishing a correspondence between $\operatorname{Ass}_R M$ and $\operatorname{Ass}_{\bar{R}} M$.

(2a): $\operatorname{Ass}_R M$ is a finite set because its cardinality equals that of $\operatorname{Ass}_{\bar{R}} M$.

(2b): We need to show that each zero-divisor of M is in some associated prime of M. Let $r \in R$, $0 \neq z \in M$, such that $rz = 0$. Then $r \in \operatorname{Ann}_R z$, so $\bar{r} \in \operatorname{Ann}_{\bar{R}} z$, so $\bar{r} \in \bar{\mathfrak{p}}$ for some $\bar{\mathfrak{p}} \in \operatorname{Ass}_{\bar{R}} M$ by the Proposition, and hence $r \in \mathfrak{p} \in \operatorname{Ass}_R M$.

The proof of the Corollary in the notes needn't be changed to work in this context. ☐

5. (a) Suppose R is a ring such that $R_{\mathfrak{m}}$ is Noetherian for every maximal ideal $\mathfrak{m}$, and such that for any $0 \neq f \in R$, only finitely many maximal ideals of R contain f. Then R is Noetherian.

   (b) Let S be a ring which is the union of a countable chain of subrings $S_1 \subset S_2 \subset S_3 \subset \cdots$. Let $\{P_n\}_{n \geq 1}$ be a countable set of prime ideals of S such that $P_n \cap S_m = 0$ whenever $n > m$. Let I be an ideal of S with $I \subseteq \bigcup_n P_n$. Show that there is some integer j such that $I \subseteq P_j$.

   (c) Conclude that the ring R from the first footnote of the Oct 17 lecture notes is Noetherian and has infinite Krull dimension.[2]

Solution. (a) Let I be an ideal of R. If $I = 0$ then I is obviously finitely generated. If $I \neq 0$, pick $0 \neq f \in I$. Let $S = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ be the set of maximal ideals of R that contain f. Then there is some integer n and some $g_{ij} \in I$ such that for $1 \leq i \leq t$, we have $(g_{i1}, \ldots, g_{in})R_{\mathfrak{m}_i} = IR_{\mathfrak{m}_i}$, since $R_{\mathfrak{m}_i}$ is Noetherian. Let J be the sub-ideal of I generated by f and all the g_{ij}. Then J is finitely generated, and we will show that $J = I$.

Let $\mathfrak{m}$ be a maximal ideal of R. If $\mathfrak{m} \notin S$, then $\frac{f}{1}$ is a unit of $R_{\mathfrak{m}}$, so $R_{\mathfrak{m}} = fR_{\mathfrak{m}} \subseteq JR_{\mathfrak{m}} \subseteq IR_{\mathfrak{m}} \subseteq R_{\mathfrak{m}}$, so all are equalities, and in particular $IR_{\mathfrak{m}} = JR_{\mathfrak{m}}$. If $\mathfrak{m} \in S$, then $\mathfrak{m} = \mathfrak{m}_i$ for some $1 \leq i \leq t$, and $JR_{\mathfrak{m}} \subseteq IR_{\mathfrak{m}} = (g_{i1}, \ldots, g_{in})R_{\mathfrak{m}} \subseteq JR_{\mathfrak{m}}$, so again $IR_{\mathfrak{m}} = JR_{\mathfrak{m}}$. Since $IR_{\mathfrak{m}} = JR_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$, it follows that $I = J$, so I is finitely generated and R is Noetherian.

(b) Again we may assume that $I \neq 0$. The conditions imply immediately that for each m, $I \cap S_m \subseteq \bigcup_{n=1}^{m} (P_n \cap S_m)$, so by Prime Avoidance in S_m, there is some n_m such that $I \cap S_m \subseteq P_{n_m} \cap S_m$. Let $M = \min\{m \mid I \cap S_m \neq 0\}$. Let $m \geq M$, and $n \in \mathbb{N}$ such that $I \cap S_m \subseteq P_n \cap S_m$. If $n > M$, then

$$0 \neq I \cap S_M = I \cap S_M \cap S_m \subseteq P_n \cap S_M \cap S_m = 0 \cap S_m = 0,$$

---

[2] *Hint:* $S_n := K[X_1, \ldots, X_{\binom{n+1}{2}}]$. Calculate S_{P_n}, and use the Hilbert Basis Theorem as well as parts (a) and (b) of this problem

which is a contradiction. Thus, for any n with $I \cap S_m \subseteq P_n \cap S_m$, we must have $n \leq M$. In particular, each $n_m \leq M$. Thus, we have

$$\begin{aligned} I = \bigcup_m (I \cap S_m) &\subseteq \bigcup_m (P_{n_m} \cap S_m) \subseteq \bigcup_m \left(\bigcup_{t \leq M} (P_t \cap S_m) \right) \\ &= \bigcup_{t \leq M} \left(\bigcup_m (P_t \cap S_m) \right) = \bigcup_{n \leq M} P_n \end{aligned}$$

Then by prime avoidance in S, $I \subseteq P_n$ for some $n \leq M$.

(c) Let S, W, P_n, and R be as in the footnote. For each n we have that P_nR is a prime ideal of R and $R_{P_nR} = (W \cup (S \setminus P_n))^{-1}S = (S \setminus P_n)^{-1}S = S_{P_n}$, since $W \subseteq S \setminus P_n$. But S_{P_n} is the localization of $T_n = k(X_1, \dots, X_{\binom{n}{2}}, X_{\binom{n+1}{2}+1}, \dots)[X_{\binom{n}{2}+1}, \dots, X_{\binom{n+1}{2}}]$ at the image of P_n, since localizing S at P_n makes X_i invertible for all $i \in \mathbb{N}_{\geq 1} \setminus \left[\binom{n}{2}+1, \binom{n+1}{2}\right]$. T_n is Noetherian by the Hilbert basis theorem, hence so are all its localizations, and in particular R_{P_nR}.

Let $S_n := k[X_1, X_2, \dots, X_{\binom{n+1}{2}}]$ for each $n \geq 1$. Then S is the union of the chain $S_1 \subset S_2 \subset \cdots$ of subrings, and clearly $P_n \cap S_m = 0$ whenever $n > m$. Now suppose $\mathfrak{m}$ is a maximal ideal of R. Then $\mathfrak{m} = W^{-1}\mathfrak{q}$ for some prime ideal $\mathfrak{q}$ of S with $\mathfrak{q} \cap W = \emptyset$. Then $\mathfrak{q} \subseteq \bigcup_n P_n$, so by part (b), $\mathfrak{q} \subseteq P_n$ for some n. Thus, $\mathfrak{m} = W^{-1}\mathfrak{q} \subseteq W^{-1}P_n = P_nR$, and since P_nR is a proper ideal of R and $\mathfrak{m}$ is maximal, we have $\mathfrak{m} = P_nR$.

Hence, $\Omega(R) = \{P_nR \mid n \geq 1\}$, and R_{P_nR} is Noetherian for each n. Finally, let $0 \neq f \in R$. Then $f = \frac{g}{w}$ for some $0 \neq g \in S$ and $w \in W$. Then $g \in S_n$ for some n, so for $m > n$ we have $P_m \cap S_n = 0$ and hence $g \notin P_m$. If $f \in P_m$ for some $m > n$, then for some $w' \in W$ we have $w'g \in P_m$, but $w' \notin P_m$, so $g \in P_m$ which is a contradiction. Thus, f is in only finitely many of the ideals P_iR. Then by part (a), R is Noetherian.

□