Exercises due Wednesday, Oct 24, 2007

Recall that an R-module N is *finitely presented* if there *exists* a short exact sequence

$$R^t \to R^s \to N \to 0 \tag{1}$$

with s, t positive integers.

1. (a) Let $L \to M \to N \to 0$ be a right-exact sequence of R-modules. Show that if L, N are finitely generated, so is M, and that if M is finitely generated, so is N.

   (b) Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules such that M is finitely generated and N is finitely presented. Show that L is finitely generated.[1]

   (c) Conclude that for any finitely presented R-module N and any surjection $\pi : R^s \twoheadrightarrow N$, $\operatorname{Ker} \pi$ is finitely generated.[2]

2. Show that if R is a Noetherian ring with an embedded prime, then R is not reduced. Is the converse true? (Prove or give a counterexample.)

3. Let M be a k-module, where k is a field. Without using Hopkins' theorem (or anything that depends on it), show that$M$ is Artinian if and only if M is a finite-dimensional vector space over k.

4. Prove that in Prop 1, Prop 2, and the final Corollary from the Oct 8 lecture notes, the conclusions hold if all the assumptions on R and M are replaced with the single assumption that M is a Noetherian module.[3]

5. (a) Suppose R is a ring such that $R_{\mathfrak{m}}$ is Noetherian for every maximal ideal $\mathfrak{m}$, and such that for any $0 \neq f \in R$, only finitely many maximal ideals of R contain f. Then R is Noetherian.

   (b) Let S be a ring which is the union of a countable chain of subrings $S_1 \subset S_2 \subset S_3 \subset \cdots$. Let $\{P_n\}_{n \geq 1}$ be a countable set of prime ideals of S such that $P_n \cap S_m = 0$ whenever $n > m$. Let I be an ideal of S with $I \subseteq \bigcup_n P_n$. Show that there is some integer j such that $I \subseteq P_j$.

   (c) Conclude that the ring R from the first footnote of the Oct 17 lecture notes is Noetherian and has infinite Krull dimension.[4]

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[1] *Hint:* Construct a right exact sequence which includes free modules and maps to the original short exact sequence, using the fact that free modules are projective, and use the Snake Lemma.

[2] We've been tacitly assuming this fact for a while.

[3] *Hint:* Let $\bar{R} := R/\operatorname{Ann} M$; M is also an $\bar{R}$-module. Start by establishing a correspondence between $\operatorname{Ass}_R M$ and $\operatorname{Ass}_{\bar{R}} M$.

[4] *Hint:* $S_n := K[X_1, \ldots, X_{\binom{n+1}{2}}]$. Calculate S_{P_n}, and use the Hilbert Basis Theorem as well as parts (a) and (b) of this problem