Solutions for Oct 8, 2007 exercises

1. Let $F : {}_R\mathbf{Mod} \to {}_T\mathbf{Mod}$ and $G : {}_T\mathbf{Mod} \to {}_R\mathbf{Mod}$ be additive functors, such that F is left-adjoint to G. Prove that G is left-exact.

   Solution. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ be a left-exact sequence of T-modules. Let φ be the adjoint map. Since left-exactness is independent of isomorphism classes, we may assume that $L \subseteq M$ and f is the inclusion map. Note that for any R-module A and T-module B, $\varphi_{A,B} : \operatorname{Hom}_T(FA, B) \to \operatorname{Hom}_R(A, GB)$ is not just a bijection, but a group isomorphism. Hence, $\varphi(0) = 0$ and $\varphi^{-1}(0) = 0$. Also, on modules we have $G(0) = 0$, since $H := \operatorname{Hom}_R(G(0), G(0)) \cong \operatorname{Hom}_T(FG(0), 0) = 0$, but H must include both the zero map and the identity map, hence the identity map is zero.

   So we want to show that the sequence

   $$0 \to GL \xrightarrow{Gf} GM \xrightarrow{Gg} GN \tag{1}$$

   is exact.

   First, note that $G(g) \circ G(f) = G(g \circ f) = G(0) = 0$ since G is an additive functor. Hence (1) is a complex.

   To see that Gf is injective, let $K = \operatorname{Ker} Gf$, so that $0 \to K \xrightarrow{i} GL \xrightarrow{Gf} GM$ is exact. Then $Gf \circ i = 0$, so $0 = \varphi^{-1}(Gf \circ i) = f \circ \varphi^{-1}(i)$, but since f is injective it follows that $\varphi^{-1}(i) = 0$, so that $i = \varphi(\varphi^{-1}(i)) = 0$. Thus, $K = 0$.

   Finally, we must show that $\operatorname{Ker} Gg \subseteq \operatorname{Im} Gf$. Let $z \in \operatorname{Ker} Gg$, $J = Rz$, and $j : J \hookrightarrow GM$ the inclusion map. Then $Gg \circ j = 0$, so $0 = \varphi^{-1}(Gg \circ j) = g \circ \varphi^{-1}(j)$, so that $\operatorname{Im} \varphi^{-1}(j) \subseteq \operatorname{Ker} g = \operatorname{Im} f = L$. That is, there is a map $t : FJ \to L$ such that $\varphi^{-1}(j) = f \circ t$. Then $j = \varphi(\varphi^{-1}(j)) = \varphi(f \circ t) = Gf \circ \varphi(t)$, so $z = j(z) = (Gf \circ \varphi(t))(z) \in \operatorname{Im} Gf$.[1] □

2. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a sequence of R-modules; call it $\mathcal{E}$. Show that the following conditions are equivalent:

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[1] An alternate way to do this problem is to show that $\operatorname{Hom}_T(FR, -)$ is left-exact, and then use the fact that this functor is essentially the same as $\operatorname{Hom}_R(R, G-)$, which is isomorphic to the functor G. But I think this way ends up being harder to make precise.

(a) $\mathcal{E}$ is an exact complex, and there is an R-linear map $p : M \to L$ such that $p \circ f = 1_L$.

(b) $\mathcal{E}$ is an exact complex, and there is an R-linear map $j : N \to M$ such that $g \circ j = 1_N$.

(c) There exist $p : M \to L$ and $j : N \to M$ such that $p \circ f = 1_L$, $g \circ j = 1_N$, and $(f \circ p) + (j \circ g) = 1_M$.

(d) There is an isomorphism $\varphi : M \xrightarrow{\sim} L \oplus N$ in such a way that $\varphi \circ f = i$ and $\pi \circ \varphi = g$, where $i : L \to L \oplus N$ is the natural inclusion and $\pi : L \oplus N \to N$ is the natural projection.

Solution (partial). (a) $\implies$ (b): Define $j : N \to M$ as follows: For $n \in N$, there is some $m \in M$ with $g(m) = n$. Then let $j(n) = m - f(p(m))$. Then show that this definition is well-defined (independent of choice of m), and then we have $g(j(n)) = g(m - fp(m)) = g(m) - gfp(m) = g(m) - 0 = n$.

(b) $\implies$ (a): Define $p : M \to L$ as follows. For $m \in M$, we have $m - jg(m) \in \operatorname{Ker} g = \operatorname{Im} f$, so there is a unique $l \in L$ such that $f(l) = m - jg(m)$; let $p(m) :=$ this l. Then $f(l) - jgf(l) = f(l)$, so $p(f(l)) = l$.

(a) $\implies$ (c): Let j be as in the proof of ((a) $\implies$ (b)). Then for any $m \in M$, we have $f(p(m)) + j(g(m)) = f(p(m)) + (m - f(p(m))) = m$.

(c) $\implies$ (d): First, $gf = gfpf = g(1 - jg)f = gf - gjgf = gf - gf = 0$. By symmetry we also have $pj = 0$.

Define $\varphi : M \to L \oplus N$ by $\varphi(m) := (p(m), g(m))$. It is clear that φ is a homomorphism. Also, $\varphi(f(l)) = (pf(l), gf(l)) = (l, 0) = i(l)$, and $\pi(\varphi(m)) = \pi(p(m), g(m)) = g(m)$. So we need only show that φ is bijective.

If $\varphi(m) = 0$, then $p(m) = g(m) = 0$, so $m = fp(m) + jg(m) = f(0) + j(0) = 0$. Finally, given any $(l, n) \in L \oplus N$, we have $\varphi(f(l) + j(n)) = (pf(l) + pj(n), gf(l) + gj(n)) = (l + 0, 0 + n) = (l, n)$.

(d) $\implies$ (a): Let $q : L \oplus N \twoheadrightarrow L$ be the natural projection, and let $p := q \circ \varphi$. Then $q \circ i = 1_L$, so $pf = q\varphi f = qi = 1_L$. Thus also f is injective, and $gf = \pi\varphi f = \pi i = 0$. For $z \in N$, we have $\varphi(y) = (0, z)$ for some $y \in M$, and then $g(y) = \pi(\varphi(y)) = z$, so g is surjective.

Finally, if $y \in \operatorname{Ker} g$, then $\pi(\varphi(y)) = 0$, so $\varphi(y) = i(x) = \varphi(f(x))$ for some $x \in L$, and then $y = f(x) \in \operatorname{Im} f$. So the complex is exact. $\square$

3. Use the previous problem to show that for any rings R and T, any additive functor $F : {}_R\mathbf{Mod} \to {}_T\mathbf{Mod}$, and any R-modules M and N, we have $F(M \oplus N) \cong F(M) \oplus F(N)$.

   Solution. Let $f : M \to M \oplus N$, $j : N \to M \oplus N$ be the natural inclusions, and $p : M \oplus N \to M$, $g : M \oplus N \to N$ be the natural projections. Then f, g, p, j satisfy the properties of (c) above. Since F preserves composition, addition, and identity maps, it follows that $F(f), F(g), F(p), F(j)$ satisfy the same properties. Hence since (c) implies (d), we have $F(M \oplus N) \cong F(M) \oplus F(N)$. $\square$

4. Let R be a ring, W a multiplicative set, and $C(W) := \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap W = \emptyset\}$. Show there is a bijection between $\operatorname{Spec} W^{-1}R$ and $C(W)$.

   Solution. The bijection is given by extension and contraction via the localization map $\ell_W : R \to W^{-1}R$.

   Take any $\mathfrak{p} \in C(W)$. Then $\mathfrak{p} \cap W = \emptyset$, so $\mathfrak{p}^{\mathrm{e}} = W^{-1}\mathfrak{p} \neq W^{-1}R$, and in fact $\mathfrak{p}^{\mathrm{e}} \in \operatorname{Spec} W^{-1}R$, since $W^{-1}R/W^{-1}\mathfrak{p} = \bar{W}^{-1}(R/\mathfrak{p})$, which is a nonzero subring of the fraction field of $R/\mathfrak{p}$, hence an integral domain.

   On the other hand, take any $P \in \operatorname{Spec} W^{-1}R$. Then P^{c} is the image of P under the map of prime spectra associated to the localization map, so it must be prime.

   Next, note that $P = P^{\mathrm{ce}}$. By an earlier exercise, we need only show that $P \subseteq P^{\mathrm{ce}}$. Let $\frac{r}{w} \in P$. Then $\ell_W(r) = \frac{r}{1} = \frac{w}{1}\frac{r}{w} \in P$, so $r \in P^{\mathrm{c}}$, and hence $\frac{r}{w} = \frac{1}{w}\ell_W(r) \in P^{\mathrm{ce}}$. Now, if $P^{\mathrm{c}} \cap W \neq \emptyset$, then $P = P^{\mathrm{ce}} = W^{-1}R$, which is impossible because P is prime. Hence, $P^{\mathrm{c}} \in C(W)$.

   Finally, for $\mathfrak{p} \in C(W)$, we have $\mathfrak{p}^{\mathrm{ec}} = \mathfrak{p}$. By an earlier exercise we need only show that $\mathfrak{p}^{\mathrm{ec}} \subseteq \mathfrak{p}$. So let $x \in \mathfrak{p}^{\mathrm{ec}}$. Then $\frac{x}{1} \in \mathfrak{p}^{\mathrm{e}}$, so that $\frac{x}{1} = \frac{p}{w}$ for some $p \in \mathfrak{p}$ and $w \in W$. Thus there is some $w' \in W$ with $w'wx = w'p \in \mathfrak{p}$, but $w'w \in W \subseteq R \setminus \mathfrak{p}$, so $x \in \mathfrak{p}$.

   $\square$

5. Say that a multiplicatively closed set $W \subseteq R$ is *saturated* if $x, y \in R$, $xy \in W \Rightarrow x \in W$ and $y \in W$. For an arbitrary multiplicatively closed set W, define the *saturation* $\widetilde{W}$ of W to be the set $\widetilde{W} := \{v \in R \mid \exists x \in R : xv \in W\}$. Show that

   (a) $\widetilde{W}$ is a saturated multiplicative set that contains W.

   (b) For any multiplicative set V such that $W \subseteq V$, we have $\widetilde{W} \subseteq \widetilde{V}$.

   (c) If W is saturated then $W = \widetilde{W}$.

   (d) If $V \subseteq W$ are multiplicative sets, then the natural map $\varphi : V^{-1}R \to W^{-1}R$ is an isomorphism if and only if $W \subseteq \widetilde{V}$.

   (e) $\widetilde{W} = R \setminus \bigcup C(W)$.

   Solution. (a) $\widetilde{W}$ is a multiplicative set because if $x, y \in \widetilde{W}$, then there are $r, s \in R$ with $rx, sy \in W$, so $rsxy \in W$, whence $xy \in \widetilde{W}$. $\widetilde{W}$ contains W because for any $w \in W$, we have $1w = w$. $\widetilde{W}$ is saturated because if $xy \in \widetilde{W}$, then $rxy = (rx)y = (ry)x \in W$ for some $r \in R$.

   (b) If $x \in \widetilde{W}$ then $rx \in W \subseteq V$, so $x \in \widetilde{V}$.

   (c) If $x \in \widetilde{W}$, then $rx \in W$, so $x \in W$.

   (d) Suppose φ is an isomorphism, and let $w \in W$. Then since φ is surjective, for some $\frac{r}{v} \in V^{-1}R$, we have $\varphi(r/v) = 1/w$, which means that for some $w' \in W$, we have $w'(rw - v) = 0$. That is, $\varphi(\frac{wr-v}{1}) = 0$, so since φ is injective we have $\frac{rw-v}{1} = 0$ in $V^{-1}R$. Thus there is some $v' \in V$ with $v'rw = v'v \in V$, so that $w \in \widetilde{V}$.

   Conversely, suppose $W \subseteq \widetilde{V}$. If $\varphi(\frac{r}{v}) = 0$, then there is some $w \in W$ with $wr = 0$, and some $s \in R$ with $sw \in V$, so $(sw)r = 0$ implies that $\frac{r}{v} = 0$ in $V^{-1}R$. Thus φ is injective.[2] Let $\frac{r}{w} \in W^{-1}R$. There is some $s \in R$ with $sw \in V$, and $\varphi(\frac{sr}{sw}) = \frac{sr}{sw} = \frac{r}{w}$, so φ is surjective.

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[2] Several people got this problem wrong because they assumed that φ is injective by definition. This is not true. For instance, suppose $V = \{1\}$ and W is some multiplicative set that contains both 1 and a zerodivisor x, so that there is some nonzero $y \in R$ with $xy = 0$. Then $\frac{y}{1} \neq 0$ in $V^{-1}R \cong R$, but $\frac{y}{1} = 0$ in $W^{-1}R$. In this case φ can be surjective but not injective: Let $R = \mathbb{Q}[x]/(x^2)$, $V = \{1\}$, and $W = \{1, x\}$. Then φ is the natural projection map $R \twoheadrightarrow \mathbb{Q}$ which sends x to 0, and $\widetilde{V} = \mathbb{Q} \not\supseteq W$.

(e) Let $x \in \widetilde{W}$. For any $\mathfrak{p} \in C(W)$, if $x \in \mathfrak{p}$, then for some $r \in R$, we have $rx \in W$ (by definition of $\widetilde{W}$) and $rx \in \mathfrak{p}$ (since $\mathfrak{p}$ is an ideal), so $rx \in \mathfrak{p} \cap W = \emptyset$, a contradiction. Thus, $\widetilde{W} \subseteq R \setminus \bigcup C(W)$. Conversely, suppose $x \in R \setminus \bigcup C(W)$. If $\frac{x}{1}$ is contained in some maximal ideal M of $W^{-1}R$, then $x \in M^c \in C(W)$ by problem 5, which contradicts our assumption on x. Hence $\frac{x}{1}$ is a unit in $W^{-1}R$, which means that $\frac{y}{w}\frac{x}{1} = \frac{1}{1}$ for some $\frac{y}{w} \in W^{-1}R$. So for some $w' \in W$, we have $(w'y)x = w'w \in W$, so that $x \in \widetilde{W}$. Thus, $R \setminus \bigcup C(W) \subseteq \widetilde{W}$.[3]

$\square$

6. Let R be a ring, M a finitely presented R-module, and N any R-module. Prove that $\operatorname{Ass}\operatorname{Hom}_R(M, N) = \operatorname{Supp} M \cap \operatorname{Ass} N$.

Solution. Let $\mathfrak{p} \in \operatorname{Ass}\operatorname{Hom}_R(M, N)$. Say $\mathfrak{p} = (0 : f)$ for some $f : M \to N$. For any $a \in \operatorname{Ann} M$ and any $z \in M$, we have $af(z) = f(az) = f(0) = 0$, so $a \in (0 : f) = \mathfrak{p}$. That is, $\mathfrak{p} \in V(\operatorname{Ann} M) = \operatorname{Supp} M$ since M is finitely generated. Say $M = \sum_{i=1}^{n} Rz_i$. Suppose that for each i, there is some $w_i \in (0 : f(z_i)) \setminus \mathfrak{p}$. Letting $w = \prod_i w_i \in R \setminus \mathfrak{p}$, we have $wf(z_i) = 0$ for all i, so $wf(M) = 0$, which means that $w \in (0 : f) \in \mathfrak{p}$, a contradiction. So for some i, we have $\mathfrak{p} = (0 : f(z_i)) \in \operatorname{Ass} N$.

Conversely, suppose $\mathfrak{p} \in \operatorname{Supp} M \cap \operatorname{Ass} N$. Let $f : M_{\mathfrak{p}} \twoheadrightarrow M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ be the natural surjection. Since $M_{\mathfrak{p}} \neq 0$, Nakayama's lemma shows that $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \neq 0$, and the latter is a finite-dimensional k-vector space, where $k = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Hence, there is a k-linear (hence $R_{\mathfrak{p}}$-linear) surjection $g : M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \twoheadrightarrow k$. So we get a nonzero $R_{\mathfrak{p}}$-linear map $g \circ f : M_{\mathfrak{p}} \to k = (R/\mathfrak{p})_{\mathfrak{p}}$, and thus by localization of Hom there is a nonzero R-linear map $v : M \to R/\mathfrak{p}$ such that $g \circ f = v_{\mathfrak{p}}$. Since $\mathfrak{p} \in \operatorname{Ass} N$, there is an injection $i : R/\mathfrak{p} \hookrightarrow N$. Let $u = i \circ v$. Since u factors through $R/\mathfrak{p}$ it is clear that $\mathfrak{p}u = 0$. On the other hand, suppose $au = 0$. Then $i \circ (av) = 0$, so i injective implies that $av = 0$. Since v is nonzero, there is some $z \in M$ with $v(z) \neq 0$. Since $av(z) = 0$, it follows that $\bar{a} = 0$ in the integral domain $R/\mathfrak{p}$, *i.e.* $a \in \mathfrak{p}$. Hence, $(0 : u) = \mathfrak{p} \in \operatorname{Ass}\operatorname{Hom}_R(M, N)$. $\square$

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[3] Thanks to William for this proof, which is simpler than the one I had in mind.