Exercises due Monday, Oct 8, 2007

1. Let $F : {}_R\mathbf{Mod} \to {}_T\mathbf{Mod}$ and $G : {}_T\mathbf{Mod} \to {}_R\mathbf{Mod}$ be additive functors, such that F is left-adjoint to G. Prove that G is left-exact.[1] (Note: It is also true that F is right-exact.)

2. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a sequence of R-modules; call it $\mathcal{E}$. Show that the following conditions[2] are equivalent:

   (a) $\mathcal{E}$ is an exact complex, and there is an R-linear map $p : M \to L$ such that $p \circ f = 1_L$.

   (b) $\mathcal{E}$ is an exact complex, and there is an R-linear map $j : N \to M$ such that $g \circ j = 1_N$.

   (c) There exist $p : M \to L$ and $j : N \to M$ such that $p \circ f = 1_L$, $g \circ j = 1_N$, and $(f \circ p) + (j \circ g) = 1_M$.

   (d) There is an isomorphism $\varphi : M \xrightarrow{\sim} L \oplus N$ in such a way that $\varphi \circ f = i$ and $\pi \circ \varphi = g$, where $i : L \to L \oplus N$ is the natural inclusion and $\pi : L \oplus N \to N$ is the natural projection.

3. Use the previous problem to show that for any rings R and T, any additive functor $F : {}_R\mathbf{Mod} \to {}_T\mathbf{Mod}$, and any R-modules M and N, we have $F(M \oplus N) \cong F(M) \oplus F(N)$. (Note: We actually get an isomorphism at the functor level.)

4. Let R be a ring, W a multiplicative set, and $C(W) := \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap W = \emptyset\}$. Show there is a bijection between $\operatorname{Spec} W^{-1}R$ and $C(W)$. (Note: It's actually a homeomorphism.)

5. Say that a multiplicatively closed set $W \subseteq R$ is *saturated* if $x, y \in R$, $xy \in W \Rightarrow x \in W$ and $y \in W$. For an arbitrary multiplicatively closed set W, define the *saturation* $\widetilde{W}$ of W to be the set $\widetilde{W} := \{v \in R \mid \exists x \in R : xv \in W\}$. Show that

   (a) $\widetilde{W}$ is a saturated multiplicative set that contains W.

[1] *Hint:* Whenever you have an element $z \in M$ of a module M, think of it in terms of the injection map $Rz \hookrightarrow M$.

[2] A sequence satisfying any or all of these conditions is called a *split short exact sequence*, and p (or j) is called a *splitting*. These play a central role in homological algebra.

(b) For any multiplicative set V such that $W \subseteq V$, we have $\widetilde{W} \subseteq \widetilde{V}$.

(c) If W is saturated then $W = \widetilde{W}$.

(d) If $V \subseteq W$ are multiplicative sets, then the natural map $\varphi : V^{-1}R \to W^{-1}R$ is an isomorphism if and only if $W \subseteq \widetilde{V}$.

(e) $\widetilde{W} = R \setminus \bigcup C(W)$.[3]

6. Let R be a ring, M a finitely presented R-module, and N any R-module. Prove that $\operatorname{Ass} \operatorname{Hom}_R(M, N) = \operatorname{Supp} M \cap \operatorname{Ass} N$.

[3] *Hint:* The proof I know of this uses Zorn's lemma, along with showing that a certain ideal J is prime. You should use ideals of the form (x), $J + (y)$ and $(J : y)$.