Exercises due Monday, Sep 24, 2007

1. Let $\mathscr{C}$ be a category.

   (a) Assume that there is a *product* functor (i.e. a right-adjoint to the diagonal functor) $\Pi : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$. Use the notation $\Pi(X, Y) = X \times Y$ and $\Pi(f, g) = f \times g$ on objects and morphisms. Prove that for every pair $Y, Z \in \operatorname{Ob}\mathscr{C}$, there is a pair of morphisms (called *projections*) $\pi_Y : Y \times Z \to Y$, $\pi_Z : Y \times Z \to Z$ such that for every object X of $\mathscr{C}$ and every pair of morphisms $f : X \to Y$, $g : X \to Z$, there is a unique morphism $\ulcorner fg \urcorner : X \to Y \times Z$ such that $\pi_Y \circ \ulcorner fg \urcorner = f$ and $\pi_Z \circ \ulcorner fg \urcorner = g$.

   (b) Conversely, assume that for every pair of objects Y, Z of $\mathscr{C}$, there is an object $Y \times Z$ and a pair of morphisms $\pi_Y : Y \times Z \to Y$, $\pi_Z : Y \times Z \to Z$, such that for every pair of morphisms $f : X \to Y$, $g : X \to Z$, there is a unique morphism $\ulcorner fg \urcorner : X \to Y \times Z$ such that $\pi_Y \circ \ulcorner fg \urcorner = f$ and $\pi_Z \circ \ulcorner fg \urcorner = g$. Prove that there is a functor $\Pi : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, whose value on objects is given by $\Pi(Y, Z) := Y \times Z$, which is a product functor for $\mathscr{C}$.

2. Show that the standard definitions for products give product functors on the categories **Set**, **CRng**, **Group**, and ${}_R$**Mod**. Show that disjoint union is (the object function for) a coproduct in the category of sets. What is the coproduct in the category ${}_R$**Mod**?[1]

3. The forgetful functor U : **Top** $\to$ **Set** has both a left adjoint ("free functor") *and* a right adjoint ("cofree functor"). What are they? Prove that they have the required properties.

4. Prove all the statements which the posted Sep 7 lecture notes ask you to prove.

5. Prove the colon identities from the Sep 12 lecture and the extension/contraction identities from the Sep 14 lecture.

6. Carefully show that any nonzero ring R has a nonempty set of *minimal* primes $\operatorname{Min} R$ (suitably defined). That is, $R = 0 \iff \operatorname{Min} R = \emptyset$. For

[1] For this exercise, you needn't consider functors $\mathscr{C}^S \to \mathscr{C}$ for general sets, but only functors $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$.

a fixed proper ideal I, use the projection $R \to R/I$ to draw a conclusion about "the minimal primes of I".

7. Let M be an R-module, and let $\mathrm{S} := \{(0 :_R z) \mid 0 \neq z \in M\}$. Prove that any maximal element of S is a prime ideal.

8. Prove the following equivalences $\mathbf{Ab} \simeq {}_{\mathbb{Z}}\mathbf{Mod}$, $\mathbf{CRng} \simeq {}_{\mathbb{Z}}\mathbf{Alg}$ of categories, where $\mathbb{Z}$ is the ring of rational integers.