Math 614: Lecture notes

Oct 1, 2007

Today we prove Nakayama-type lemmas ("almost zero" = "zero", for finitely generated modules), and a rather simple structure theorem for finitely generated flat modules over a local ring.

Nakayama-type lemmas

Proposition 1 (The Determinant Trick). *Let M be an n-generated R-module (i.e. finitely generated by n elements), let I be an ideal, and let $\varphi : M \to M$ be an R-linear map such that $\varphi(M) \subseteq IM$. Then there is an equation in $\operatorname{Hom}_R(M, M)$ of the form*

$$\varphi^n + a_1\varphi^{n-1} + \cdots + a_{n-1}\varphi + a_n \cdot 1_M = 0_M,$$

where $a_j \in I^j$ for $1 \le j \le n$.

Proof. Say $M = \sum_{i=1}^n Rz_i$. Then there exist $r_{ij} \in I$, $1 \le i \le n$, $1 \le j \le n$, such that $\varphi(z_j) = \sum_{i=1}^n r_{ij}z_i$. In other words, if we consider the matrix $A = (r_{ij})$ and $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$, the $n \times n$ identity matrix, we have

$$(\varphi I_n - A) \cdot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let $B := \varphi I_n - A$, and let B' be the adjoint of the matrix of cofactors of B. Then from linear algebra we have that $B'B = (\operatorname{Det} B)I_n$, where 'Det' is the usual determinant. In particular, we have $(\operatorname{Det} B)(z_i) = 0$ for $1 \leq i \leq n$, so that $(\operatorname{Det} B)1_M = 0_M$ in $\operatorname{Hom}_R(M, M)$. However, expanding out the determinant of B, we get an alternating sum of terms, where each term is a product of n elements, where for each such term, say j of them are φ and the other $n-j$ of them are of the form a_{kl}. Hence we get an equation of the required form, since each $a_{kl} \in I$ and so a product of $n-j$ of them will sit in I^{n-j}.

[For example if $n = 2$, then

$$\begin{aligned}\operatorname{Det} B = \operatorname{Det}\begin{pmatrix}\varphi - a_{11} & -a12 \\ -a21 & \varphi - a_{22}\end{pmatrix} &= (\varphi - a_{11})(\varphi - a_{22}) - a_{12}a_{21} \\ &= \varphi^2 - (a_{11} + a_{22})\varphi + (a_{11}a_{22} - a_{12}a_{21}).\end{aligned}$$

] □

Corollary (Nakayama lemma, version 1). *If M is finitely generated and $M = IM$, then there is some $a \in I$ with $1 + a \in \operatorname{Ann} M$. Hence, if $I \subseteq \bigcap \Omega(R)$ (the Jacobson radical of R), then $M = 0$.*

Proof. For the first statement, let $\varphi = 1_M$ and $a = a_1 + \cdots + a_n$ in the Determinant Trick above.

For the second statement, if $a \in I$ then $a \in \mathfrak{m}$ for all $\mathfrak{m} \in \Omega(R)$, which means that $1 + a$ is in no maximal ideal of R, and hence is a unit. Thus, a unit annihilates M, which implies that $M = 0$. □

Corollary (Nakayama lemma, version 2). *If $N \subseteq M$ are R-modules such that M/N is finitely generated, $J = \bigcap \Omega(R)$, and $M = N + JM$, then $M = N$.*

Proof. Apply version 1 to M/N. We have $M/N = (N + JM)/N = J(M/N)$, so that $M/N = 0$ by Version 1, which means that $M = N$. □

Definition. A subset S of an R-module M is a *minimal generating set* for M if (1) S generates M, and (2) No proper subset of S generates M as an R-module.

Note that if M is finitely generated, one can obtain a minimal generating set from an arbitrary finite generating set by throwing out unneccessary generators one at a time.

Proposition 2. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, k)$, and $z_1, \dots, z_n \in M$. Then*

1. *The z_i generate M $\iff$ their images generate $M/\mathfrak{m}M$ as a k-vector space.*

2. *Suppose the z_i generate M. Then they form a* minimal *generating set for M $\iff$ their images form a* basis *for the vector space $M/\mathfrak{m}M$ $\iff$ $\pi \otimes_R 1_k : R^n \otimes k \to M \otimes k$ is an isomorphism, where $\pi : R^n \to M$ is the map which sends each $e_i \mapsto z_i$.*

Proof.

1. The forward implication is clear, since we have surjections $R^n \twoheadrightarrow M \twoheadrightarrow M/\mathfrak{m}M$, and the composition factors through the surjection $R^n \twoheadrightarrow k^n$, so that we get $k^n \twoheadrightarrow M/\mathfrak{m}M$.

   Now suppose that the $\overline{z_i}$ generate $M/\mathfrak{m}M$, and let $N = \sum_{i=1}^n Rz_i$. Then $M = N + \mathfrak{m}M$, so by Version 2 of the Nakayama lemma, $M = N$.

2. If the images don't form a basis, then some $\overline{z_i}$ (without loss of generality $i = n$) is a k-linear combination of the other z_j. Hence, $\overline{z_1}, \dots, \overline{z_{n-1}}$ is a set of generators for $M/\mathfrak{m}M$, so by part 1, $z_1, \dots, z_{n-1}$ generate M, so that the original set of generators is not minimal.

   If the images do form a basis and $(\pi \otimes_R 1_k)(\sum_i \overline{r_i} \otimes e_i) = 0$, then $\sum_i \overline{r_i z_i} = 0$, so that each $\overline{r_i} = 0$, which means that $\sum_i \overline{r_i} \otimes e_i = 0$. Hence, $\pi \otimes_R 1_k$ is injective, hence an isomorphism.

   Finally, if the generating set is not minimal, then some z_i (without loss of generality $i = n$) is a linear combination of the other z_j, say $z_n = \sum_{i=1}^{n-1} r_i z_i$. Then $(r_1, \dots, r_{n-1}, -1) \in \operatorname{Ker} \pi$, so we have that $0 \neq (\overline{r_1}, \dots, \overline{r_{n-1}}, \overline{-1}) \in \operatorname{Ker}(\pi \otimes_R 1_k)$, hence $\pi \otimes_R 1_k$ is not an isomorphism. □

Proposition 3. *If $(R, \mathfrak{m}, k)$ is a local ring and L, M are nonzero finitely generated R-modules, then $L \otimes_R M \neq 0$.*

Proof. By the Nakayama lemma, version 1, we have $\bar{L} := L/\mathfrak{m}L = 0$ and $\bar{M} := M/\mathfrak{m}M \neq 0$. Hence, they are vector spaces over k of positive dimension, so we have $\dim_k(\bar{L} \otimes_k \bar{M}) = \dim_k(\bar{L}) \cdot \dim_k(\bar{M}) \neq 0$. Moreover, by what we know of tensor products, we have

$$L \otimes_R M \twoheadrightarrow \bar{L} \otimes_R M \twoheadrightarrow \bar{L} \otimes_R \bar{M} \cong \bar{L} \otimes_k \bar{M} \neq 0,$$

and hence $L \otimes_R M \neq 0$. □

Corollary. *If L, M are finitely generated modules over a ring R, then*

$$\operatorname{Supp}(L \otimes_R M) = \operatorname{Supp} L \cap \operatorname{Supp} M.$$

Proof. We already know the '$\subseteq$' direction. On the other hand, let $\mathfrak{p} \in \operatorname{Supp} L \cap \operatorname{Supp} M$. Then $(L \otimes_R M)_{\mathfrak{p}} \cong L_{\mathfrak{p}} \otimes_{\mathfrak{p}} M_{\mathfrak{p}}$, and since $R_{\mathfrak{p}}$ is local and $L_{\mathfrak{p}}$, $M_{\mathfrak{p}}$ are nonzero finitely-generated $R_{\mathfrak{p}}$-modules, the previous corollary implies that $L_{\mathfrak{p}} \neq 0$ and $M_{\mathfrak{p}} \neq 0$, so that $\mathfrak{p} \in \operatorname{Supp} L \cap \operatorname{Supp} M$. ☐

Lemma (The Snake Lemma). *Suppose the following diagram of R-modules commutes and has exact rows:*

$$\begin{tikzcd}
 & L \arrow[r, "\alpha"] \arrow[d, "f"] & M \arrow[r, "\beta"] \arrow[d, "g"] & N \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' &
\end{tikzcd}$$

Then there is an exact sequence

$$\operatorname{Ker} f \xrightarrow{\tilde{\alpha}} \operatorname{Ker} g \xrightarrow{\tilde{\beta}} \operatorname{Ker} h \xrightarrow{\delta} \operatorname{Cok} f \xrightarrow{\overline{\alpha'}} \operatorname{Cok} g \xrightarrow{\overline{\beta'}} \operatorname{Cok} h.$$

I won't prove this lemma; it's a standard exercise in homological algebra. The maps $\tilde{\alpha}$ and $\tilde{\beta}$ are restrictions of α and β respectively, and the maps $\overline{\alpha'}$ and $\overline{\beta'}$ are induced by α' and β' respectively. The trick is defining δ, which is obtained by a "diagram chase" from N to L' (*Hint:* Start with the fact that β is surjective..) Then it's a good calisthenic to show that everything is well-defined and the sequence is exact.

Proposition 4. *Let N be a finitely presented module over a local ring $(R, \mathfrak{m})$. Then the following are equivalent:*

a. N is free.

b. N is projective.

c. N is flat.

d. The map $i : \mathfrak{m} \otimes_R N \to N$ which sends each $a \otimes z \mapsto az$ is injective.

Proof. We already know the implications $a \Rightarrow b \Rightarrow c$, and the implication $c \Rightarrow d$ follows because i is essentially the map $j \otimes_R 1_N$, where $j : \mathfrak{m} \hookrightarrow R$ is the inclusion, and flatness means that tensoring with N preserves the injectiveness of *any* R-module injection.

So it remains to prove $d \Rightarrow a$: Assume that i is injective, let $z_1, \dots, z_n$ be a minimal generating set of N, and consider the short exact sequence

$$0 \to L \to R^n \xrightarrow{\pi} N \to 0,$$

where the surjection is defined by $e_i \mapsto z_i$. Then L is finitely generated (since N is finitely presented). We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & \mathfrak{m} \otimes_R L \arrow[r] \arrow[d, "f:=j\otimes 1_L"] & \mathfrak{m} \otimes_R R^n \arrow[r, "1_{\mathfrak{m}}\otimes\pi"] \arrow[d, "g:=j\otimes 1_{R^n}"] & \mathfrak{m} \otimes_R N \arrow[r] \arrow[d, "h:=j\otimes 1_N"] & 0 \\
0 \arrow[r] & L \arrow[r] & R^n \arrow[r, "\pi"] & N \arrow[r] & 0
\end{tikzcd}$$

Then by the Snake Lemma, we have an exact sequence

$$\begin{tikzcd}
\operatorname{Ker} h \arrow[r] \arrow[d, equal] & \operatorname{Cok} f \arrow[r] \arrow[d, equal] & \operatorname{Cok} g \arrow[r] \arrow[d, equal] & \operatorname{Cok} h \arrow[d, equal] \\
0 \arrow[r] & L/\mathfrak{m}L \arrow[r] & R^n \otimes_R k \arrow[r, "\bar{\pi}", "\cong"'] & N \otimes_R k
\end{tikzcd}$$

We have $\operatorname{Ker} h = 0$ by assumption, and $\bar{\pi}$ is an isomorphism by Proposition 2. Hence, $L/\mathfrak{m}L$ is the kernel of an isomorphism, which means that $L/\mathfrak{m}L = 0$, so that $L = \mathfrak{m}L$, and then by Version 1 of the Nakayama lemma we have $L = 0$. Thus, π is an isomorphism, which means that $N \cong R^n$ is free. $\square$