Math 614: Lecture notes

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The following lemma is of independent interest. It is called "prime avoidance" because it implies that if an ideal 'avoids' each of a finite set of primes, then it 'avoids' their union as well.

Proposition (Prime Avoidance lemma). *Let R be a ring, I an ideal and let $Q_1, \ldots, Q_n$ be a finite set of ideals, all but possibly two of which are prime, such that for all $1 \leq j \leq n$, $I \nsubseteq Q_j$. Then $I \nsubseteq \bigcup_{j=1}^{n} Q_j$.*

Proof. There is nothing to prove if $n = 0, 1$, so we may assume $n \geq 2$. By induction on n, I is not contained in a union of a strictly smaller number of the Q_j. By renumbering if necessary, $Q_3, \ldots, Q_n$ are prime. Let $a_j \in (Q_j \cap I) \setminus \bigcup_{i \neq j} Q_i$. (By minimality of the cover, such a_j's exist.) Let

$$b := \prod_{\{t \geq 3 | a_1 + a_2 \notin Q_t\}} a_t$$

(where the empty product is 1) and

$$a := a_1 + a_2 + a_1 a_2 b.$$

Clearly $a = a_1(1 + a_2 b) + a_2 \notin Q_1$ and $a = a_1 + a_2(1 + a_1 b) \notin Q_2$. Also, for $t \geq 3$, if $a_1 + a_2 \in Q_t$ then the factors of b are not in Q_t, so that $a_1 a_2 b \notin Q_t$ since Q_t is prime, and hence $a \notin Q_t$. If on the other hand $a_1 + a_2 \notin Q_t$, then $a_t \in Q_t$ is a factor of b, which means that $b \in Q_t$, so that $a \notin Q_t$. Thus, $a \in I \setminus \bigcup_{j=1}^{n} Q_j$, as required. □

Corollary. *Let R be Noetherian and M a finitely generated R-module. Then the map*

$$\varphi : M \to \bigoplus_{\mathfrak{p} \in \operatorname{Ass} M} M_{\mathfrak{p}}$$

given by the direct product of the localization maps, is injective. Thus, $M = 0 \iff M_{\mathfrak{p}} = 0$ *for all* $\mathfrak{p} \in \operatorname{Ass} M$*. If* $\ell(M) < \infty$*, then* φ *is an isomorphism.*

Proof. Let $\operatorname{Ass} M = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, where $\mathfrak{p}_i \neq \mathfrak{p}_j$ for $i < j$. If $\varphi(z) = 0$, then for all $\mathfrak{p} \in \operatorname{Ass} M$ we have that $(0 : z) \not\subseteq \mathfrak{p}$, so by Prime Avoidance we have $(0 : z) \not\subseteq \bigcup \operatorname{Ass} M$. Since this union is the set of zerodivisors of M, there is some nonzerodivisor w of M such that $wz = 0$. Hence $z = 0$.

If $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Ass} M$, then φ is the zero map, but it is injective, so $M = 0$.

Suppose $\ell(M) < \infty$. Then $R/\operatorname{Ann} M$ is Artinian, so the support of M matches the assassinator of M, a finite set of maximal ideals. Take any $\mathfrak{m} \in \Omega(R)$. If $\mathfrak{m} \notin \operatorname{Ass} M$, then $\mathfrak{m} \notin \operatorname{Supp} M$, so $M_{\mathfrak{m}} = (M_{\mathfrak{p}})_{\mathfrak{m}} = 0$ for any $\mathfrak{p} \in \operatorname{Spec} R$, and hence $\varphi_{\mathfrak{m}} : 0 \to 0$ is the identity map in this case. If $\mathfrak{m} \in \operatorname{Ass} M$, then for each component $M_{\mathfrak{p}}$ of the direct sum, we have $(M_{\mathfrak{p}})_{\mathfrak{m}} = 0$ if $\mathfrak{p} \neq \mathfrak{m}$ and $(M_{\mathfrak{p}})_{\mathfrak{m}} = M_{\mathfrak{m}}$ if $\mathfrak{p} = \mathfrak{m}$. So again we have that $\varphi_{\mathfrak{m}} : M_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is the identity map. Hence, since it is locally an isomorphism at maximal ideals, we have that φ is an isomorphism. ☐

Corollary. *An Artinian ring is a direct product of finitely many local Artinian local rings.*

Proof. By the above, we have $R \cong \bigoplus_{\mathfrak{p} \in \operatorname{Ass} R} R_{\mathfrak{p}} = \prod_{\mathfrak{p} \in \operatorname{Ass} R} R_{\mathfrak{p}}$, and each $R_{\mathfrak{p}}$ is an Artinian local ring since it has only one prime ideal, $\mathfrak{p}R_{\mathfrak{p}}$. ☐

Primary decomposition

In finitely generated modules over Noetherian rings, there is an analogue of the theory of unique factorization. We represent any submodule $N \subseteq M$ as an *intersection* of "primary" submodules of M, in a way which is in a certain sense unique. Namely, for each prime $\mathfrak{p} \in \operatorname{Ass}(M/N)$, one gets a "$\mathfrak{p}$-primary component", which if $\mathfrak{p}$ is minimal in $\operatorname{Ass}(M/N)$ is uniquely determined by M/N.

Definition. Let $N \subseteq M$ be a submodule and $\mathfrak{p} \in \operatorname{Spec} R$.

We say N is a $\mathfrak{p}$*-primary* submodule of M if $\operatorname{Ass} M/N = \{\mathfrak{p}\}$. If $\operatorname{Ass} M = \{\mathfrak{p}\}$, we say M is a $\mathfrak{p}$*-coprimary* module. A submodule N of M is *primary* if it is $\mathfrak{p}$-primary for some $\mathfrak{p} \in \operatorname{Spec} R$.

We say N is an *irreducible* submodule if whenever N', N'' are submodules of M and $N = N' \cap N''$, either $N = N'$ or $N = N''$.

Any representation of a submodule $N \subseteq M$ as an intersection

$$N = N_1 \cap \cdots \cap N_r$$

of a finite number of submodules N_i of M is called a *decomposition* of N. The decomposition is *irredundant* if for any $1 \leq i \leq r$, $N \neq \bigcap_{i \neq j} N_i$. If each of the N_i is irreducible, we call the decomposition *irreducible.* If each N_i is primary, we call it a *primary* decomposition. In an irredundant primary decomposition, if each N_i is $\mathfrak{p}_i$-primary, and $\mathfrak{p}_i \neq \mathfrak{p}_j$ for all $i \neq j$, we call the decomposition a *minimal primary decomposition* of N, and we say N_i is the *P-primary component* of the decomposition.

Note: If N, N' are $\mathfrak{p}$-primary submodules of M, so is $N \cap N'$. This is because one can think of $M/(N \cap N')$ as a submodule of $\frac{M}{N} \oplus \frac{M}{N'}$, via the short exact sequence

$$0 \to \frac{M}{N \cap N'} \to \frac{M}{N} \oplus \frac{M}{N'} \to \frac{M}{N + N'} \to 0.$$

Hence, any irredundant primary decomposition can be replaced with a minimal primary decomposition, by grouping those terms which are $\mathfrak{p}$-primary for each $\mathfrak{p}$.

Theorem (Primary decomposition). *Let R be a Noetherian ring and M a finitely generated R-module.*

(a) Any submodule of M has an irreducible decomposition.

(b) Any irreducible submodule of M is primary. Hence, any submodule has a primary decomposition.

(c) If $N = \bigcap_{i=1}^{r} N_i$, with each N_i $\mathfrak{p}_i$-primary, is an irredundant primary decomposition of a proper submodule $N \subset M$, then $\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

(d) Suppose $N = \bigcap_{i=1}^{r} N_i$ is a minimal primary decomposition, where each N_i is $\mathfrak{p}_i$-primary, and $W \subset R$ is a multiplicative set such that $\mathfrak{p}_i \cap W = \emptyset$ for $1 \leq i \leq t$ and $\mathfrak{p}_i \cap W \neq \emptyset$ for $i > t$. Then $W^{-1}N = \bigcap_{i=1}^{t} W^{-1}N_i$ is a minimal primary decomposition of $W^{-1}N$ in $W^{-1}M$, where each $W^{-1}N_i$ is $W^{-1}\mathfrak{p}_i -$ primary.

(e) *If $N \subset M$ is a proper submodule and $\mathfrak{p}$ is a* minimal *element of* $\operatorname{Ass}(M/N)$*, then the $\mathfrak{p}$-primary component of N in any shortest primary decomposition of N is $\ell_{\mathfrak{p}}^{-1}(N_{\mathfrak{p}})$, where $\ell_{\mathfrak{p}} : M \to M_{\mathfrak{p}}$ is the localization map.*

Proof. (a) If not, let $\mathcal{S}$ be the set of submodules of M having no irreducible decomposition. Let N be a maximal element of $\mathcal{S}$. Then N is not irreducible, so $N = N_1 \cap N_2$, where $N \neq N_1$ and $N \neq N_2$. But N_1 and N_2 properly contain N, so by maximality they each have an irreducible decomposition, and gluing these together gives an irreducible decomposition of N.

(b) Suppose N is not primary. Then there are two distinct primes $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Ass}(M/N)$. Then there are submodules $R/\mathfrak{p}_i \cong L_i/N \subseteq M/N$. And for any $z \in L_i \backslash N$, we have $\mathfrak{p}_i = (N : z)$, which means that $L_1 \cap L_2 = N$, so N is not irreducible.

(c) We may assume $N = 0$, and that the decomposition is minimal. M is isomorphic to a submodule of $\bigoplus_{i=1}^r M/N_i$, so

$$\operatorname{Ass} M \subseteq \bigcup_{i=1}^{r} \operatorname{Ass}(M/N_i) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}.$$

On the other hand, $\mathfrak{p}_1 \in \operatorname{Ass} M$. To see this, let $B := \bigcap_{j=2}^r N_j$, which is nonzero by minimality of the decomposition. Observe that

$$0 \neq B = B/0 = \frac{B}{N_1 \cap B} \cong \frac{B + N_1}{N_1} \subseteq M/N_1.$$

Hence, $\emptyset \neq \operatorname{Ass} B \subseteq \operatorname{Ass}(M/N_1) = \{\mathfrak{p}_1\}$, so $\mathfrak{p}_1 \in \operatorname{Ass} B \subseteq \operatorname{Ass} M$. Repeating the argument for each i, $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} \subseteq M$.

(d) Since localization commutes with intersection, we get the equality. Moreover, for $i > t$ we have $\operatorname{Ass} W^{-1}(M/N_i) = \emptyset$, so $W^{-1}M = W^{-1}N_i$ and these terms are removable from the intersection. For $i \leq t$, $\operatorname{Ass}(W^{-1}M/W^{-1}N_i) = \operatorname{Ass} W^{-1}(M/N_i) = \{W^{-1}\mathfrak{p}_i\}$, as claimed, and all these prime ideals are distinct.

(e) Say $\mathfrak{p} = \mathfrak{p}_1$. If $\frac{z}{1} \in N_{\mathfrak{p}}$, then $az \in N \subseteq N_1$ for some $a \in R \setminus \mathfrak{p}$, but a is a nonzerodivisor on M/N_1, so $z \in N_1$.

Conversely, $N_{\mathfrak{p}} = (N_1)_{\mathfrak{p}}$ by part (d), since for all $j \geq 2$, we have $\mathfrak{p}_j \not\subseteq \mathfrak{p}$. □

So, the primary submodules corresponding to *minimal* primes in a primary decomposition are unique, but the others rarely are.

For example, let $R = k[x, y]$ where k is a field, and $I = (x^2, xy)$. Then $(x^2, xy) = (x) \cap (x^2, y)$ is a primary decomposition of I in R, but so is $(x^2, xy) = (x) \cap (x^2, xy, y^n)$ for all $n \geq 2$. This behavior is typical.

Definition. If $\mathfrak{p}$ is a prime ideal and n is a positive integer, the *nth symbolic power* of $\mathfrak{p}$, denoted $\mathfrak{p}^{(n)}$ is the $\mathfrak{p}$-primary component of $\mathfrak{p}^n$ in a minimal primary decomposition of R. Specifically, $\mathfrak{p}^{(n)} = (\mathfrak{p}^n)^{\mathrm{ec}}$, where extension and contraction is in terms of the localization map $R \to R_{\mathfrak{p}}$.