

Math 614: Lecture notes

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Some dimension theory

Definition. Let R be a ring. A *chain* of prime ideals is a finite sequence of proper containments

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

where each $\mathfrak{p}_i \in \text{Spec } R$. The chain above has *length* n . We say the chain is *saturated* if there no proper refinement can be given by inserting another prime ideal. The (*Krull*) *dimension* $\dim R$ of R is defined to be the supremum of the lengths of all chains of prime ideals of R .

The *height* of a prime ideal \mathfrak{p} is defined to be $\text{ht } \mathfrak{p} := \dim R_{\mathfrak{p}}$.¹ For an arbitrary proper ideal I of R , we define $\text{ht } I := \inf\{\text{ht } \mathfrak{p} \mid I \subseteq \mathfrak{p} \in \text{Spec } R\}$. The *dimension* of an ideal I is $\dim I := \dim R/I$.

If M is an R -module, we define $\dim M := \dim R/\text{Ann } M$.

We say that R is *catenary* if any containment $\mathfrak{p} \subsetneq \mathfrak{q}$ of prime ideals has a saturated refinement, and every such refinement of the given pair has the same length.

There are several things we can say immediately from the definitions:

- (a) If R is an integral domain (or more generally if R has a unique minimal prime \mathfrak{p}), then $\dim R$ is the supremum of the chains of primes where the smallest prime is \mathfrak{p} .
- (b) If (R, \mathfrak{m}) is local, then $\dim R$ is the supremum of the chains where the largest prime is \mathfrak{m} .

¹Equivalently, $\text{ht } \mathfrak{p}$ is the supremum of lengths of chains of prime ideals where the biggest prime in the chain is \mathfrak{p} .

(c) For any ideal I , we have the inequality

$$\text{ht } I + \dim R/I \leq \dim R.$$

To see this, first assume $I = \mathfrak{p}$ is prime, and let $h = \text{ht } \mathfrak{p}$ and $d = \dim R/\mathfrak{p}$. If $h = \infty$ (resp. $d = \infty$), then the chains of primes contained in \mathfrak{p} (resp. containing \mathfrak{p}) are unbounded, and hence the chains of primes in R are unbounded. If both numbers are finite, then there exist chains of primes

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_h = \mathfrak{p}$$

corresponding to a chain in $R_{\mathfrak{p}}$, and

$$\mathfrak{p} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_d$$

corresponding to a chain in R/\mathfrak{p} , so that gluing these chains together we get a chain in R of length $d + h$. Hence, the supremum of all chains in R is $\geq d + h$.

In the general case, let \mathfrak{p} be a minimal prime over I such that $\dim R/\mathfrak{p} = \dim R/I$. Then

$$\dim R \geq \text{ht } \mathfrak{p} + \dim R/\mathfrak{p} = \text{ht } \mathfrak{p} + \dim R/I \geq \text{ht } I + \dim R/I$$

(d) Similarly if $I \subseteq \mathfrak{q}$ where \mathfrak{q} is a prime ideal, then $\text{ht } \mathfrak{q} \geq \text{ht } I + \text{ht}(\mathfrak{q}/I)$. This follows from the other inequality by localizing at \mathfrak{q} .

Beyond these things, there is not much we can say about height and dimension for ideals and modules over an arbitrary ring. However, if R is *Noetherian*, we can characterize height in terms of *number of generators up to radical!*

Dimension in Noetherian rings

For this section, we make the standing assumption that R is *Noetherian*.

Theorem (Krull's Principal Ideal Theorem (PIT)). *For any $x \in R$ and any prime ideal \mathfrak{p} which is minimal over (x) , $\text{ht } \mathfrak{p} \leq 1$.*

Proof. First we prove it in the case where (R, \mathfrak{p}) is a local ring. We may assume without loss of generality that $\text{ht } \mathfrak{p} > 0$, so that there is some prime \mathfrak{q} with $\mathfrak{q} \subsetneq \mathfrak{p}$. Since \mathfrak{p} is minimal over x , it follows that $x \notin \mathfrak{q}$. Moreover, $\bar{\mathfrak{p}}$ is minimal over (0) in the ring $R/(x)$, and hence $R/(x)$ is Artinian. Thus, the following descending chain of ideals cannot be strictly descending:

$$\mathfrak{q} + (x) \supseteq \mathfrak{q}^{(2)} + (x) \supseteq \mathfrak{q}^{(3)} + (x) \supseteq \dots$$

Hence, there is some n with $\mathfrak{q}^{(n)} + (x) = \mathfrak{q}^{(n+1)} + (x)$. Now take any $f \in \mathfrak{q}^{(n)}$. Then $f = g + xr$ for some $g \in \mathfrak{q}^{(n+1)}$ and $r \in R$. So we have $xr = f - g \in \mathfrak{q}^{(n)}$, so that in $R_{\mathfrak{q}}$ we have $\frac{xr}{1} \in \mathfrak{q}^{(n)} R_{\mathfrak{q}}$, but $x \notin \mathfrak{q}$ means that $\frac{x}{1}$ is a unit in $R_{\mathfrak{q}}$, so that $\frac{r}{1} \in \mathfrak{q}^{(n)} R_{\mathfrak{q}}$ and $r \in \mathfrak{q}^{(n)}$. Thus, $f \in x\mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)}$.

We have shown that $\mathfrak{q}^{(n)} = x\mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)}$, so that by the Nakayama lemma we have that $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. Now apply the Nakayama lemma in $R_{\mathfrak{q}}$ to conclude that $\mathfrak{q}^{(n)} = 0$. Hence, \mathfrak{q} is minimal over the zero ideal, which implies that $\text{ht } \mathfrak{q} = 0$. Since $\mathfrak{q} \subsetneq \mathfrak{p}$ was arbitrary, $\text{ht } \mathfrak{p} = 1$.

In the general case, $\mathfrak{p}R_{\mathfrak{p}}$ is minimal over $xR_{\mathfrak{p}}$, so $\text{ht } \mathfrak{p} = \text{ht } \mathfrak{p}R_{\mathfrak{p}} \leq 1$. \square

Essentially as a corollary, we get the much more powerful-looking next theorem:

Theorem (Krull's height theorem). *Let R be a Noetherian ring.*

- (a) *If $\mathfrak{p} \subseteq \mathfrak{q}$ are two prime ideals in R , there is a saturated chain of prime ideals starting with \mathfrak{p} and ending with \mathfrak{q} .*
- (b) *If $x_1, \dots, x_n \in R$ and $\mathfrak{p} \in \text{Spec } R$ is minimal over (x_1, \dots, x_n) , then $\text{ht } \mathfrak{p} \leq n$. Hence, $\text{ht } \mathfrak{p} < \infty$.*
- (c) *If $\mathfrak{p} \in \text{Spec } R$ with $\text{ht } \mathfrak{p} = n$, then there exist elements $x_1, \dots, x_n \in \mathfrak{p}$ such that \mathfrak{p} is minimal over (x_1, \dots, x_n) .*

Proof. (a) If $\mathfrak{p} = \mathfrak{q}$, the single element comprises a saturated chain. Otherwise, do the following inductive process. Say we have a saturated chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_i$$

of primes, with $\mathfrak{p}_i \subsetneq \mathfrak{q}$. Then pick $x \in \mathfrak{q} \setminus \mathfrak{p}_i$, and choose a prime \mathfrak{p}_{i+1} such that in R/\mathfrak{p}_i , $\mathfrak{p}_{i+1}/\mathfrak{p}_i$ is minimal over $(x) + \mathfrak{p}_i$. Then $\text{ht}(\mathfrak{p}_{i+1}/\mathfrak{p}_i) = 1$, so the enlarged chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_i \subset \mathfrak{p}_{i+1}$$

is also saturated.

Since R is Noetherian, this process can't proceed forever, so $\mathfrak{p} = \mathfrak{p}_n$ for some n , and we get the desired saturated chain.

(b) First suppose (R, \mathfrak{p}) is local.

We go by induction on n , where the $n = 1$ case is PIT.

Let $\mathfrak{q} \subsetneq \mathfrak{p}$ be the top of a saturated chain of primes that ends with \mathfrak{p} . Then not all the x_i are in \mathfrak{q} , so without losing generality $x_n \notin \mathfrak{q}$. Then \mathfrak{p} is minimal over $\mathfrak{q} + (x_n)$, so that some power of \mathfrak{p} is contained in $\mathfrak{q} + (x_n)$. Hence, there is some s such that for $1 \leq i < n$ we have equations

$$x_i^s = y_i + x_n r_i, \quad y_i \in \mathfrak{q}, \quad r_i \in R.$$

Hence, we have $\sqrt{(x_1, \dots, x_n)} = \sqrt{(y_1, \dots, y_{n-1}, x_n)}$, so that \mathfrak{p} is minimal over $(y_1, \dots, y_{n-1}, x_n)$.

If $(y_1, \dots, y_{n-1}) \subseteq \mathfrak{q}' \subseteq \mathfrak{q}$ and \mathfrak{q}' is prime, then \mathfrak{p} is minimal over $\mathfrak{q}' + (x)$, and hence by the PIT, $\text{ht}(\mathfrak{p}/\mathfrak{q}') \leq 1$. Since we already have $\mathfrak{q} \subsetneq \mathfrak{p}$, it follows that $\mathfrak{q} = \mathfrak{q}'$. Hence, \mathfrak{q} is minimal over (y_1, \dots, y_{n-1}) . Then by induction on n , $\text{ht} \mathfrak{q} \leq n - 1$. Since this is true whenever \mathfrak{q} is the second-to-last step in a saturated primes ending in \mathfrak{p} , it follows that $\text{ht} \mathfrak{p} \leq n$.

In the general case, $\mathfrak{p}R_{\mathfrak{p}}$ is minimal over $(x_1, \dots, x_n)R_{\mathfrak{p}}$, so that $\text{ht} \mathfrak{p} = \text{ht} \mathfrak{p}R_{\mathfrak{p}} \leq n$.

The last statement follows because there is a finite generating set $\mathfrak{p} = (x_1, \dots, x_n)$, and then $\text{ht} \mathfrak{p} \leq n$.

(c) Let $x_1 \in \mathfrak{p} \setminus \bigcup \text{Min} R$, $x_2 \in \mathfrak{p} \setminus \bigcup \text{Min}(R/(x_1))$, and so forth. As long as \mathfrak{p} is not minimal over (x_1, \dots, x_i) , we can pick $x_{i+1} \in \mathfrak{p} \setminus \bigcup \text{Min}(R/(x_1, \dots, x_i))$ because of Prime Avoidance.

For $1 \leq r \leq n$, we have $\text{ht}(x_1, \dots, x_{r-1}) = r - 1 < \text{ht} \mathfrak{p}$ by induction, so we can construct x_r as above. Since x_r is not in any of the minimal primes over (x_1, \dots, x_{r-1}) , we have $\text{ht}(x_1, \dots, x_r) \geq r$, but also $\text{ht}(x_1, \dots, x_r) \leq r$ by part (b). Hence, $\text{ht}(x_1, \dots, x_n) = n = \text{ht} \mathfrak{p}$, so that \mathfrak{p} is minimal over (x_1, \dots, x_n) . □