Math 614: Lecture notes

Oct 3, 2007

The main topic for today is that of "local properties". But before we go into it, the student may at this point be under the impression that the notions of 'free', 'projective', and 'flat' are equivalent for modules over a ring. This is not true, as the following examples show:

The $\mathbb{Z}$-module $\mathbb{Q}$ is flat but not projective. It is flat because it is a localization of $\mathbb{Z}$. On the other hand, if it is projective, it has to be the direct summand of a free $\mathbb{Z}$-module – i.e. of a free abelian group. So suppose there is a free abelian group F and a pair of group homomorphisms $F \underset{j}{\overset{\pi}{\rightleftarrows}} \mathbb{Q}$ such that $\pi \circ j = 1_{\mathbb{Q}}$. Then we have

$$j(1) = \sum_{i=1}^{r} a_i e_i,$$

where $\{e_1, \ldots, e_r\}$ is part of a fixed free basis of F and $a_i \in \mathbb{Z}$. Let $m := 1 + \max_i |a_i|$. Then $mj(1/m) = j(1) = \sum_{i=1}^{r} a_i e_i$. Hence, the freeness of the basis implies that m divides each a_i, but then $|a_i| < m$ implies that each $a_i = 0$. That is, $j(1) = 0$, so that $\pi(j(1)) = \pi(0) = 0 \neq 1$, and $\pi \circ j \neq 1_{\mathbb{Q}}$.

As for the other supposed implication, if $R = A \times B$ is a product of nonzero rings, then the R-module $A = R/B$ is projective but not free. It is projective because it is clearly a direct summand of R. On the other hand, since the nonzero R-ideal B kills A, it follows that A is not R-free.

Local properties

First we note the following exercise-worthy facts, given without proof, about any flat base change.

Proposition. *Let F be a flat R-module, and let L, M be R-submodules of N. We may think of $F \otimes_R L$ as a submodule of $F \otimes_R N$, and similarly for the other submodules of N. With this identification,*

1. *If $L \subseteq M$, then $F \otimes L \subseteq F \otimes M$.*
2. *$F \otimes (L + M) = (F \otimes L) + (F \otimes M)$.*
3. *$F \otimes (L \cap M) = (F \otimes L) \cap (F \otimes M)$.*
4. *If $f : M \to Q$ is an R-linear homomorphism, then $\operatorname{Ker}(1_F \otimes_R f) = F \otimes_R \operatorname{Ker} f$ as submodules of M, and $\operatorname{Im}(1_F \otimes_R f) = F \otimes_R \operatorname{Im} f$ as submodules of Q.*
5. *$S \otimes_R F$ is a flat S-module for any R-algebra S.*

Proposition (Some Local Properties). *(a) Let $N \subseteq M$ and $u \in M$. If $\frac{u}{1} \in N_P$ for all $P \in \Omega(R)$, then $u \in N$.*

(b) Let $N \subseteq M$. If $N_P = M_P$ for all $P \in \Omega(R)$ then $N = M$.

(c) Let $\mathcal{S}$ be a sequence of R-module homomorphisms. If $\mathcal{S}_P$ is a complex (resp. exact) for all $P \in \Omega(R)$, then so is $\mathcal{S}$. Hence the property of being a kernel, a cokernel, or an image can be checked locally at maximal ideals.

(d) Let L, M be submodules of N. If $L_P \subseteq M_P$ for all $P \in \Omega(R)$, then $L \subseteq M$.

(e) An R-module F is flat if F_P is flat over R_P for all $P \in \Omega(R)$.

(f) For any multiplicative set W, $W^{-1}\mathcal{N}(R) = \mathcal{N}(W^{-1}R)$.

(g) R is reduced $\iff$ $W^{-1}R$ is reduced for all multiplicative sets W $\iff$ R_P is reduced $\forall P \in \Omega(R)$.

Proof. (a): If $u \notin N$, then there is some maximal ideal P with $P \supseteq \operatorname{Ann}\left(\frac{N+Ru}{N}\right) =: I$. If $\frac{u}{1} \in N_P$, then for some $w \in R \setminus P$, we have $wu \in N$, and hence $w(N + Ru) \subseteq N$, so that $w \in I \subseteq P$, which is a contradiction. So $\frac{u}{1} \notin N_P$.

(b): If $N \neq M$, apply (a) to any $u \notin N$.

(c): By repeated application, we need only consider sequences of the form $\mathcal{S} : L \xrightarrow{f} M \xrightarrow{g} N$. We fix such an $\mathcal{S}$.

First suppose each $\mathcal{S}_P$ is a complex for all $P \in \Omega(R)$. Then for all such P:

$$(\operatorname{Im}(g \circ f))_P = \operatorname{Im}((g \circ f)_P) = \operatorname{Im}(g_P \circ f_P) = \operatorname{Im} 0 = 0,$$

so that $\operatorname{Im}(g \circ f) = 0$, whence $g \circ f = 0$.

Now suppose each $\mathcal{S}_P$ is exact. We know that $\mathcal{S}$ is a complex, so let $x \in \operatorname{Ker} g$. Then for each $P \in \Omega(R)$:

$$\frac{x}{1} \in (\operatorname{Ker} g)_P = \operatorname{Ker} g_P = \operatorname{Im} f_P = (\operatorname{Im} f)_P,$$

so that by part (a), $x \in \operatorname{Im} f$. Hence, $\mathcal{S}$ is exact.

(d): We have

$$0 = \frac{L_P}{N_P \cap L_P} = \frac{L_P}{(N \cap L)_P} = \left(\frac{L}{N \cap L}\right)_P$$

for each $P \in \Omega(R)$, so that $L = N \cap L$, which means that $L \subseteq N$.

(e): Let $j : L \hookrightarrow M$ be an R-module injection, and consider the exact sequence

$$0 \to K \to F \otimes_R L \xrightarrow{j' := 1 \otimes j} F \otimes_R M.$$

Localizing at any $P \in \Omega(R)$, we get

$$\begin{array}{ccccccc}
0 \longrightarrow & K_P & \longrightarrow & (F \otimes_R L)_P & \xrightarrow{j'_P} & (F \otimes_R M)_P \\
& \| & & \downarrow \cong & & \downarrow \cong \\
0 \longrightarrow & K_P & \longrightarrow & F_P \otimes_{R_P} L_P & \xhookrightarrow{1 \otimes j_P} & F_P \otimes_{R_P} M_P
\end{array}$$

Since j is injective, so is j_P since R_P is flat over R, and since F_P is flat over R_P, $1_{F_P} \otimes_{R_P} j_P$ is also injective. Hence, $K_P = 0$ for all $P \in \Omega(R)$, from which it follows that $K = 0$. Thus, j' is injective, so F is flat over R.

(f): If $\frac{x}{w} \in W^{-1}\mathcal{N}(R)$, then without loss of generality $x^t = 0$. Hence, $(x/w)^t = x^t/w^t = 0$, so that $\frac{x}{w} \in \mathcal{N}(W^{-1}R)$. Conversely, suppose that $\frac{x}{w} \in \mathcal{N}(W^{-1}R)$. Then for some positive integer t, we have $x^t/w^t = (x/w)^t = 0$. Then for some $v \in W$ we have $vx^t = 0$, so that $(vx)^t = 0$ as well. Thus, $vx \in \mathcal{N}(R)$, which means that $x/w = vx/vw \in W^{-1}\mathcal{N}(R)$.

(g): Suppose R is reduced. Then for any multiplicative set W, we have $\mathcal{N}(W^{-1}R) = W^{-1}\mathcal{N}(R) = W^{-1}0 = 0$. If R_P is reduced for all $P \in \Omega(R)$, then $0 = \mathcal{N}(R_P) = \mathcal{N}(R)_P$ for all such P, and hence $\mathcal{N}(R) = 0$.

$\square$

Now we present a 'global' version of the last proposition from the Oct 1 lecture:

Theorem. *Let F be a finitely presented R-module. The following are equivalent:*

(a) F is projective.

(b) F is flat.

(c) F is 'locally free', in the sense that for all $\mathfrak{p} \in \operatorname{Spec} R$, $F_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-free.

(d) (The same as (c), but for $\mathfrak{p} \in \Omega(R)$.)

Proof. We already know that projective modules are flat. If F is flat, then $F_{\mathfrak{p}}$ is flat (hence free) over $R_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ by the proposition from Oct 1. Finally, suppose $F_{\mathfrak{p}}$ is free for all $\mathfrak{p} \in \Omega(R)$. Let $\pi : R^n \twoheadrightarrow F$ be a surjection from a finite free module. We have an exact sequence

$$\operatorname{Hom}_R(F, R^n) \xrightarrow{\pi_*} \operatorname{Hom}_R(F, F) \to C \to 0.$$

For any $\mathfrak{p} \in \Omega(R)$, the fact that localization commutes with Hom for finitely presented modules, along with the flatness of $F_{\mathfrak{p}}$, means that we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\operatorname{Hom}_R(F, R^n)_{\mathfrak{p}} & \xrightarrow{(\pi_*)_{\mathfrak{p}}} & \operatorname{Hom}_R(F, F)_{\mathfrak{p}} & \longrightarrow & C_{\mathfrak{p}} & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \| & & \\
\operatorname{Hom}_{R_{\mathfrak{p}}}(F_{\mathfrak{p}}, R_{\mathfrak{p}}^n) & \xrightarrow{(\pi_{\mathfrak{p}})_*} & \operatorname{Hom}_{R_{\mathfrak{p}}}(F_{\mathfrak{p}}, F_{\mathfrak{p}}) & \longrightarrow & 0 & &
\end{array}$$

Hence, $C_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \Omega(R)$, so that $C = 0$ and π_* is surjective. Hence, there is some $j : F \to R^n$ such that $1_F = \pi_*(j) = \pi \circ j$, so that F is a direct summand of R^n and thus projective. □

Chain conditions: Noetherian and Artinian

For any partially ordered set Σ, the following conditions are equivalent:

1. Every nonempty subset of Σ has a maximal element.

2. Any ascending chain $c_1 < c_2 < \cdots$ of elements of Σ must stop after a finite number of steps.

The proof is an easy exercise in the Axiom of Choice. Such a Σ is said to satisfy the *ascending chain condition*, or *a.c.c.*. If the order is reversed and the condition holds, we say Σ satisfies the *descending chain condition*, or *d.c.c.*.

If the submodules of an R-module M satisfy a.c.c. under the containment order, we say that M is *Noetherian*. If they satisfy d.c.c, we say M is *Artinian*. So a ring is Noetherian (resp. Artinian) if its ideals satisfy a.c.c. (resp. d.c.c.).

Both Noetherian and Artinian modules are very important. As it happens, Noetherian rings are more important than Artinian ones. In the next lecture we will see that *M is Noetherian if and only if every submodule of M is finitely generated.* After this, we can give the following examples:

1. A field is both Artinian and Noetherian, since it has only two ideals.

2. Let D be a principal ideal domain that is not a field. Then D is Noetherian but not Artinian. It is Noetherian because every ideal is finitely generated (by a single element!). On the other hand, let $x \in R$ be a nonzero non-unit. Then

$$(x) \supsetneq (x^2) \supsetneq (x^3) \supsetneq \cdots$$

is an infinite strictly descending chain of ideals. (Strict because for any n, $(x^n)/(x^{n+1}) \cong D/(x) \neq 0$.) Hence D is not Artinian.