Math 614: Lecture notes

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Chain conditions, cont'd

When talking about chains of submodules and finiteness thereof, the following notion is useful.

Definition. By definition, a *finite filtration* (of *length* n) of a module M is a sequence of R-modules containments of the form

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M, \tag{1}$$

If $M_i/M_{i-1} \neq 0$ for all $0 \leq i \leq n$, we say that the filtration is *proper*.

In the following theorem, we collect together the main important points to understand, from the beginning, about Noetherian and Artinian modules.

Theorem 1. *Let R be a ring and let L, M, N be R-modules. Let $\mathcal{P}$ be either of the properties "Noetherian" or "Artinian".*

(a) M satisfies $\mathcal{P}$ $\iff$ every submodule of M satisfies $\mathcal{P}$.

(b) M is Noetherian $\iff$ every submodule of M is finitely generated.

(c) If $\mathcal{S} : 0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is exact, then M satisfies P $\iff$ both L and N satisfy $\mathcal{P}$.

(d) Given any filtration of M as in (1)*, the following are equivalent:*

i. M satisfies $\mathcal{P}$.

ii. M_k/M_{k-1} satisfies $\mathcal{P}$ for all $1 \leq k \leq n$.

iii. M_k *satisfies* $\mathcal{P}$ *for all* $0 \leq k \leq n$.

Hence, if $N_1, \ldots, N_n$ *are R-modules, then* $\bigoplus_{i=1}^n N_i$ *satisfies* $\mathcal{P}$ *iff each* N_i *satisfies* $\mathcal{P}$.

(e) If a finitely generated R-module M *satisfies* $\mathcal{P}$*, then the ring* $R/\operatorname{Ann}(M)$ *satisfies* $\mathcal{P}$.

(f) If R *satisfies* $\mathcal{P}$ *and* M *is finitely generated, then* M *satisfies* $\mathcal{P}$.

Proof. (a) If M is Noetherian (resp. Artinian) and N is a submodule of M, then every nonempty set of submodules of N, since it is also a nonempty set of submodules of M, has a maximal (resp. minimal) element. Hence N is Noetherian (resp. Artinian). The reverse implication follows because M is a submodule of itself.

(b) If some submodule N of M is not finitely generated, let $\{z_1, z_2, \ldots\}$ be a countably infinite subset of a minimal set of generators for N. Then

$$Rz_1 \subsetneq Rz_1 + Rz_2 \subsetneq Rz_1 + Rz_2 + Rz_3 \subsetneq \cdots$$

is an infinite strictly ascending chain of submodules of M. So M is not Noetherian.

Conversely, suppose every submodule of M is finitely generated. Let

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$$

be a chain of submodules of M, and let $N := \bigcup_i N_i$. Then N is finitely generated; say $N = \sum_{i=1}^r Rz_i$. Then for each i, there is some N_{k_i} with $z_i \in N_{k_i}$, and without loss of generality $k_1 \leq k_2 \leq \cdots \leq k_r =: k$. Then $z_1, \ldots, z_r \in N_k$, but they generate N. Hence, $N \subseteq N_k \subseteq N_j \subseteq N$ (so that all three modules are equal) for all $j \geq k$. So M is Noetherian.

(c) First, note that for any submodules $X \subseteq Y$ of M, if $\beta(X) = \beta(Y)$ and $\alpha^{-1}(X) = \alpha^{-1}(Y)$, then $X = Y$. To see this, let $y \in Y$. Then $\beta(y) = \beta(x)$ for some $x \in X$, and then $y - x \in \operatorname{Ker}\beta = \operatorname{Im}\alpha$, so that $\alpha(w) = y - x$. Then $w \in \alpha^{-1}(Y) = \alpha^{-1}(X)$, so that $y - x = \alpha(w) \in X$. Hence, $y \in X$, so $X = Y$.

If M is Noetherian (resp. Artinian), then we already know that L has the corresponding chain condition, and since submodules of N correspond bijectively to submodules of M that contain L, then every nonempty subset of submodules of N has a maximal (resp. minimal) element, which means that N has the corresponding chain condition.

As for the other direction, suppose L and N are Artinian, and let $\mathcal{X} : X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$ be a descending chain of submodules of M. Then $\alpha^{-1}(\mathcal{X})$ is a descending chain of submodules of L and $\beta(\mathcal{X})$ is a descending chain of submodules of N, so there is some positive integer k such that $\alpha^{-1}(X_i) = \alpha^{-1}(X_k)$ and $\beta(X_i) = \beta(X_k)$ for all $i \geq k$. Since we also have $X_i \subseteq X_k$, the first paragraph of this proof implies that $X_i = X_k$. Hence M is Artinian.

The proof for the Noetherian case is identical; merely reverse the direction of all the containments.

(d) (i. $\Longrightarrow$ iii.) since $\mathcal{P}$ passes to submodules.

(iii. $\Longrightarrow$ ii.) by (c), since $\mathcal{P}$ passes to quotients.

(ii. $\Longrightarrow$ iii.) by induction on k: The case $k = 0$ is trivial. Suppose $k > 0$ and that M_j satisfies $\mathcal{P}$ for all $j < k$. Then the short exact sequence $0 \to M_{k-1} \to M_k \to M_k/M_{k-1} \to 0$, together with (c), shows that M_k satisfies $\mathcal{P}$.

(iii. $\Longrightarrow$ i.) because $M_n = M$.

The last statement follows by letting each $M_i := \bigoplus_{j=1}^{i} N_j$.

(e) Since M is finitely generated, say $M = \sum_{i=1}^{n} Rz_i$. Consider the R-module homomorphism $\varphi : R \to M^n$ given by $r \mapsto (rz_1, rz_2, \dots, rz_n)$. Clearly we have $\operatorname{Ann} M = \operatorname{Ker} \varphi$. We have that M^n satisfies $\mathcal{P}$ by (d), and hence its submodule $\varphi(R) \cong R/\operatorname{Ann}(M)$ also satisfies $\mathcal{P}$.

(f) There is a surjection $R^n \twoheadrightarrow M$. Since R satisfies $\mathcal{P}$, (d) shows that R^n satisfies $\mathcal{P}$, and then M satisfies $\mathcal{P}$ by (c).

□

Refinements and Length

Let

$$\mathcal{F} : 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

be a finite filtration of M. Consider another finite filtration of M:

$$\mathcal{G} : 0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_r = M.$$

If $\mathcal{F}$ and $\mathcal{G}$ are proper finite filtrations of M, we say they are *equivalent* if $n = r$ and there is some permutation σ of $\{1, \ldots, n\}$ such that $M_i/M_{i-1} \cong N_{\sigma(i)}/N_{\sigma(i)-1}$ for each $1 \leq i \leq n$. It is clear that this gives an equivalence relation on proper finite filtrations of M.

On the other hand, if $\mathcal{F}$ is proper, we say that $\mathcal{G}$ is a *refinement* of $\mathcal{F}$ if it is proper and there exist integers $0 = k_0 < k_1 < \cdots < k_{n-1} < k_n = r$ such that $N_{k_i} = M_i$ for each i.

Theorem (Schreier's refinement theorem). *If $\mathcal{F}$ and $\mathcal{G}$ above are proper finite filtrations of M, then there are refinements $\mathcal{F}'$ of $\mathcal{F}$ and $\mathcal{G}'$ of $\mathcal{G}$ which are equivalent.*

Proof. Let $M_{n+1} := N_{r+1} := M$. Then for $0 \leq i \leq n$ and $0 \leq j \leq r$, let

$$M_{i,j} := M_i + (M_{i+1} \cap N_j) \qquad \text{and} \qquad N_{j,i} := N_j + (M_i \cap N_{j+1}).$$

Note that $M_{0,0} = N_{0,0} = 0$, for $0 \leq i < n$ we have $M_{i,r} = M_{i+1} = M_{i+1,0}$, and for $0 \leq j < r$ we have $N_{j,n} = N_{j+1} = N_{j+1,0}$. Consider the filtrations:

$$\begin{array}{llcccccl}
\mathcal{F}_0 : & M_{00} & \subseteq & M_{01} & \subseteq \cdots \subseteq & M_{0r} & (= M_{10} = M_1) \\
 & & \subseteq & M_{11} & \subseteq \cdots \subseteq & M_{1r} & (= M_{20} = M_2) \\
 & & \subseteq & \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} & \\
 & & \subseteq & M_{n-1,1} & \subseteq \cdots \subseteq & M_{n-1,r} & (= M_n = M)
\end{array}$$

and

$$\begin{array}{llcccccl}
\mathcal{G}_0 : & N_{00} & \subseteq & N_{01} & \subseteq \cdots \subseteq & N_{0n} & (= N_{10} = N_1) \\
 & & \subseteq & N_{11} & \subseteq \cdots \subseteq & N_{1n} & (= N_{20} = N_2) \\
 & & \subseteq & \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} & \\
 & & \subseteq & N_{r-1,1} & \subseteq \cdots \subseteq & N_{r-1,n} & (= N_r = M).
\end{array}$$

of M. Clearly, both filtrations have length nr. Moreover, for any pair (i, j) with $0 \leq i < n$ and $0 \leq j < r$, we have

$$\begin{aligned}
\frac{M_{i,j+1}}{M_{i,j}} = \frac{M_i + (M_{i+1} \cap N_{j+1})}{M_i + (M_{i+1} \cap N_j)} &\cong \frac{M_{i+1} \cap N_{j+1}}{(M_i \cap N_{j+1}) + (M_{i+1} \cap N_j)} \\
&\cong \frac{N_j + (M_{i+1} \cap N_{j+1})}{N_j + (M_i \cap N_{j+1})} = \frac{N_{j,i+1}}{N_{j,i}}.
\end{aligned}$$

Thus, deleting the duplicate entries of $\mathcal{F}_0$ and $\mathcal{G}_0$ gives a pair of proper filtrations $\mathcal{F}'$ and $\mathcal{G}'$ of M, which are refinements of $\mathcal{F}$ and $\mathcal{G}$ respectively, and which are equivalent to each other. $\square$

A nonzero R-module M is *simple* if there are no proper submodules. It is equivalent to say that $M \cong R/\mathfrak{m}$ for some $\mathfrak{m} \in \Omega(R)$.

If every quotient M_i/M_{i-1} from a proper filtration $\mathcal{F}$ of M is simple, we call $\mathcal{F}$ a *composition series* for M.

Corollary (Jordan-Hölder theorem). *Let M be an R-module that has a composition series. Then any proper filtration of M has a refinement which is a composition series, and any two composition series of M are equivalent.*

Proof. Let $\mathcal{F}$ be a composition series and let $\mathcal{G}$ be any proper filtration of M. Then $\mathcal{F}$ and $\mathcal{G}$ have a pair of equivalent refinements (by Schreier's refinement theorem), but $\mathcal{F}$ is the only refinement of $\mathcal{F}$ since each quotient from $\mathcal{F}$ is simple. Hence, some refinement $\mathcal{G}'$ of $\mathcal{G}$ is equivalent to $\mathcal{F}$, and hence $\mathcal{G}'$ is a composition series. Note that this also holds if $\mathcal{G}$ were a composition series to begin with, so that in this case $\mathcal{G} = \mathcal{G}'$ is equivalent to $\mathcal{F}$. □

Definition. For any R-module M, define the *length* $\ell(M)$ of M as follows: If M does not have a composition series, set $\ell(M) := \infty$. Otherwise, we let $\ell(M)$ be the length of a composition series for M.[1]

Proposition. *Let R be a ring and L, M, N be R-modules.*

(a) If $0 \to L \to M \to N \to 0$ is a short exact sequence, we have

$$\ell(M) = \ell(L) + \ell(N). \tag{2}$$

(b) $\ell(M) < \infty \iff M$ is both Noetherian and Artinian.

Proof. (a) Since length is obviously constant on isomorphism classes, we may assume that $L \subseteq M$ is a submodule and $N = M/L$. Suppose first that $\ell(M) = r < \infty$. To check the equality (2), we may clearly assume that $0 \neq L$ and $L \neq M$. Then $0 \subset L \subset M$ is a proper filtration of M, which can then be refined to a composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_i = L \subset \cdots \subset M_r = M. \tag{3}$$

Then

$$0 = M_0 \subset M_1 \subset \cdots \subset M_i = L \tag{4}$$

[1] By the Jordan-Hölder Theorem, this number does not depend on the choice of composition series.

is a composition series for L, so $\ell(L) = i < \infty$. Moreover, for any $j \geq i$, we have $\frac{M_{j+1}/L}{M_j/L} \cong M_{j+1}/M_j$, so that

$$0 = M_i/L \subset M_{i+1}/L \subset \cdots \subset M_r/L = N \tag{5}$$

is a composition series for N, so that $\ell(N) = r - i = \ell(M) - \ell(L) < \infty$.

On the other hand, if $i := \ell(L) < \infty$ and $r - i := \ell(N) < \infty$, then there is some composition series of L of the form (4), and some composition series of N of the form (5), which can be glued together to get a composition series of the form (3) for M, so that $\ell(M) = r = \ell(L) + \ell(N) < \infty$.

So, $\ell(M) = \infty \iff$ either $\ell(L) = \infty$ or $\ell(N) = \infty$, and so the length equality still holds in the case where $\ell(M) = \infty$.

(b) First suppose that M is both Noetherian and Artinian, and is nonzero. By the Artinian condition, there is some simple submodule M_1 of M. If $M/M_1 \neq 0$, it is also Artinian and has a simple submodule M_2/M_1, and we can continue this process to get a chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots$$

in which every successive quotient is a nonzero simple module. Since M is Noetherian, this chain can have only finitely many steps, so we get a composition series.

Conversely, suppose $n = \ell(M) < \infty$, and proceed by induction on n to show that M is Noetherian and Artinian. If $n = 1$ then M is simple and the result is trivial. If $n > 1$ and all modules of length $< n$ are Noetherian and Artinian, let L be a simple submodule of M. Then we have $\ell(M/L) = n - 1 < n$, so by induction M/L is Noetherian and Artinian. The same is true for L, and then the short exact sequence $0 \to L \to M \to M/L \to 0$ finishes the proof.

□