Math 614: Lecture notes

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Note that some finiteness condition is required in Schreier's refinement theorem and the Jordan-Hölder theorem. For example, let $R = \mathbb{Q}[x]$, let $\mathfrak{m}_1 := (x)$, and let $\mathfrak{m}_2 := (x^2 - 2)$. It is clear that for every $n \geq 0$, we have $\mathfrak{m}_i^n / \mathfrak{m}_i^{n+1} \cong R/\mathfrak{m}_i$ for $i = 1, 2$. Also, $R/\mathfrak{m}_1 \cong \mathbb{Q}$ and $R/\mathfrak{m}_2 \cong \mathbb{Q}(\sqrt{2})$. So we have the following descending filtrations of R:

$$\mathcal{F}_i : R \supset \mathfrak{m}_i \supset \mathfrak{m}_i^2 \supset \mathfrak{m}_i^3 \supset \cdots$$

for $i = 1, 2$. Both filtrations are "composition series" in the sense that there exist no proper refinements, since every successive quotient $\mathfrak{m}_i^n / \mathfrak{m}_i^{n+1}$ is a field. However, since the field associated to $\mathcal{F}_1$ differs from that associated to $\mathcal{F}_2$, these filtrations are certainly not equivalent.

Artinian rings

It may be surprising that, although the notions of "Artinian" and "Noetherian" are independent in the case of modules, this is not the case with rings. In fact, an Artinian ring is a very special kind of Noetherian ring:

Theorem (Hopkins' Theorem). *A ring R is Artinian $\iff$ R is a Noetherian ring in which every prime ideal is maximal.*

Proof. Let R be an Artinian ring. Our first goal is to show that every prime ideal is maximal. Consider the set of ideals which are products of maximal ideals. This is a nonempty set, so it has a minimal element, say $J = \mathfrak{m}_1 \cdots \mathfrak{m}_k$. For any $\mathfrak{m} \in \Omega(R)$, $\mathfrak{m}J$ is a product of maximal ideals and $\mathfrak{m}J \subseteq J$, so by minimality $J = \mathfrak{m}J \subseteq \mathfrak{m}$. In particular, *$J$ is contained in the Jacobson radical of R.*

If $J \neq 0$, then the set of ideals which do not annihilate J is nonempty (since R is in this set), so it has a minimal element, say $\mathfrak{q}$. For some $f \in \mathfrak{q}$ we have $fJ \neq 0$, so by minimality of $\mathfrak{q}$, $\mathfrak{q} = Rf$. Also, since $J^2 \subseteq J$ is a product of maximal ideals, we have $J^2 = J$. Hence $(\mathfrak{q}J)J = \mathfrak{q}J^2 = \mathfrak{q}J \neq 0$, and $\mathfrak{q}J \subseteq \mathfrak{q}$, so by minimality of $\mathfrak{q}$ we have $\mathfrak{q}J = \mathfrak{q}$. Then by the Nakayama Lemma (since $\mathfrak{q}$ is finitely generated by f), we have $\mathfrak{q} = 0$, which annihilates J, giving the desired contradiction.

Hence $J = 0$, and so for any $\mathfrak{p} \in \operatorname{Spec} R$, we have $\mathfrak{m}_1 \cdots \mathfrak{m}_k = 0 \in \mathfrak{p}$, so that $\mathfrak{m}_j \subseteq \mathfrak{p}$ for some j, whence $\mathfrak{m}_j = \mathfrak{p}$. Thus, $\operatorname{Spec} R = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_k\}$. That is, $\operatorname{Spec} R$ is a *finite set of maximal ideals!*

Now, consider the following filtration of R:

$$\mathcal{F} : 0 = \mathfrak{m}_1 \cdots \mathfrak{m}_k \subseteq \mathfrak{m}_1 \cdots \mathfrak{m}_{k-1} \subseteq \cdots \subseteq \mathfrak{m}_1 \subseteq R.$$

Every quotient in the filtration is of the form $F_i = N/\mathfrak{m}_i N$, where $N = \mathfrak{m}_1 \cdots \mathfrak{m}_{i-1}$. This is a vector space over $R/\mathfrak{m}_i$, and is Artinian, hence it is a finite-dimensional vector space (Exercise!). Thus, each F_i has a composition series, which together refine $\mathcal{F}$ into a composition series for R. Hence, R has finite length and is thus Noetherian.

Conversely, suppose R is a Noetherian ring that is not Artinian. Let I be a maximal element in the nonempty set of ideals $\{J \mid R/J \text{ is not Artinian}\}$. Then I is prime. To see this, set $S := R/I$, and suppose $x, y \in S$ with $xy = 0$ but $x \neq 0$. The quotient of S by any proper ideal is Artinian, so since $xS \neq 0$, S/xS is both Artinian and Noetherian, and so has finite length. So if xS had finite length, we would have $\ell(S) = \ell(xS) + \ell(S/xS) < \infty$, a contradiction. Thus, since $xS \cong S/\operatorname{Ann}(x)$, we have $\ell(S/\operatorname{Ann}(x)) = \ell(xS) = \infty$, so that the condition on proper ideals of S means that $y \in \operatorname{Ann}(x) = 0$. Thus S is an integral domain, so I is prime, but clearly not maximal since $\ell(R/\mathfrak{m}) = 1$ for any $\mathfrak{m} \in \Omega(R)$. Hence, a Noetherian ring in which every prime ideal is maximal must be Artinian. □

Corollary. *If M is a finitely generated module over an Artinian ring R, then $\ell(M) < \infty$.*

Proof. R is Artinian and Noetherian by Hopkins' theorem, hence M is Artinian and Noetherian by last lecture's Theorem 1, so $\ell(M) < \infty$. □

Associated primes, revisited

Proposition 1. *Let R be a ring, $W \subseteq R$ a multiplicative set, and M an R-module*

(a) If R is Noetherian, then $M = 0 \iff \operatorname{Ass} M = \emptyset$.

(b) If M is finitely presented, then Ass *commutes with localization, in the sense that*

$$\operatorname{Ass} W^{-1}M = \{W^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass} M,\ \mathfrak{p} \cap W = \emptyset\}$$

(c) If R is Noetherian and M is finitely generated, then the minimal elements of $\operatorname{Supp} M$ *are in* $\operatorname{Ass} M$.

Proof. (a) If $M \neq 0$, then the set of proper ideals $\{(0 : z) \mid 0 \neq z \in M\}$ is nonempty, hence it has a maximal element, which by the exercise is prime, hence $\in \operatorname{Ass} M$.

(b) Suppose $\mathfrak{p} \in \operatorname{Ass} M$ with $\mathfrak{p} \cap W = \emptyset$. Then $\mathfrak{p} = (0 : z)$ for some $z \in M$, and $W^{-1}\mathfrak{p} \in \operatorname{Spec} W^{-1}R$. In fact, $W^{-1}\mathfrak{p} = \operatorname{Ann}_{W^{-1}R}(z/1)$. For if $\frac{a}{w}\frac{z}{1} = 0$, then for some $w' \in W$ we have $w'az = 0$, so $w'a \in \mathfrak{p}$ but $w' \notin \mathfrak{p}$ so $a \in \mathfrak{p}$. Conversely if $\frac{a}{w} \in W^{-1}\mathfrak{p}$, then for some $b \in \mathfrak{p}$, $w' \in W$, we have $\frac{a}{w} = \frac{b}{w'}$, so that $\frac{a}{w}\frac{z}{1} = \frac{b}{w'}\frac{z}{1} = \frac{bz}{w'} = 0$. Hence $W^{-1}\mathfrak{p} \in \operatorname{Ass} W^{-1}M$.

On the other hand, suppose $W^{-1}\mathfrak{p} \in \operatorname{Ass} W^{-1}M$. Then in particular $W^{-1}\mathfrak{p} \in \operatorname{Spec} W^{-1}R$, so that $\mathfrak{p} \cap W = \emptyset$. Also, there is an injection $j : W^{-1}R/W^{-1}\mathfrak{p} \hookrightarrow W^{-1}M$, so by Hom commuting with localization we have that $j = W^{-1}i$ for some map $i : R/\mathfrak{p} \to M$. Moreover, i is injective, because if $i(\bar{r}) = 0$, then $0 = j(\bar{r}/1)$, so that for some $w \in W$ we have $w\bar{r} = \bar{0}$, or in other words $wr \in \mathfrak{p}$. But $w \notin \mathfrak{p}$ so $r \in \mathfrak{p}$, which is to say that $\bar{r} = 0$. Thus, i is an injection, so $\mathfrak{p} \in \operatorname{Ass} M$.

(c) By the exercise, M is finitely presented. So let $\mathfrak{p}$ be a minimal element of $\operatorname{Supp} M$. Then $M_\mathfrak{p} \neq 0$, so by (b), we have

$$\begin{aligned}\emptyset \neq \operatorname{Ass} M_\mathfrak{p} &= \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Ass} M,\ \mathfrak{q} \subseteq \mathfrak{p}\} \subseteq \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Supp} M,\ \mathfrak{q} \subseteq \mathfrak{p}\} \\ &= \{\mathfrak{p}R_\mathfrak{p}\}.\end{aligned}$$

since $\mathfrak{p}$ is minimal in $\operatorname{Supp} M$. Thus, $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass} M_\mathfrak{p}$, so that $\mathfrak{p} \in \operatorname{Ass} M$, again by (b).

∎

So now we have a partial picture of $\operatorname{Ass} M$, at least in the case of a finitely generated module M over a Noetherian ring R. We know that it is nonempty when M is nonzero, its minimal elements are the minimal primes of M, and its maximal elements are exactly the ideals which are maximal with respect to being annihilators of elements of M. Moreover, membership in $\operatorname{Ass} M$ can be checked locally.

In case the reader thinks that $\operatorname{Ass} M$ always consists of the minimal primes of M, here's a counterexample:

Let $R = k[x, y]$, where k is a field and x, y are indeterminates. Let $I = (x^2, xy)$, and $M = R/I$. Then $\operatorname{Ann}_R(\bar{x}) = (I : x) = (x, y)$ and $\operatorname{Ann}_R(\bar{y}) = (I : y) = (x)$. Both of these are prime ideals, hence both are in $\operatorname{Ass} M$, but $(x) \subsetneq (x, y)$.

If $\mathfrak{p} \subsetneq \mathfrak{q}$ and both are in $\operatorname{Ass} M$, we call $\mathfrak{q}$ an *embedded prime of* M. In an *exercise*, you will show that the existence of an embedded prime of R implies that R is not reduced.

To further fill out our picture of Ass, we have the following facts:

Proposition 2. *Let R be Noetherian and M finitely generated:*

(a) $\operatorname{Ass} M$ *is a finite set.*

(b) $\bigcup \operatorname{Ass} M = \{r \in R \mid \exists z \in M, z \neq 0$ *such that* $rz = 0\}$. *In other words,* $\bigcup \operatorname{Ass} M$ *consists of all the* zero-divisors *of* M*.*

Proof. (a) If $M \neq 0$, then there exists some $\mathfrak{p}_1 \in \operatorname{Ass} M$, so that M has a submodule $M_1 \cong R/\mathfrak{p}_1$. If $M = M_1$, then we're done; otherwise, there is some $\mathfrak{p}_2 \in \operatorname{Ass}(M/M_1)$, so that for some M_2 we have $M_1 \subset M_2 \subset M$ with $M_2/M_1 \cong R/\mathfrak{p}_2$. We can continue in this way to get:

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots$$

with every containment proper and $M_i/M_{i-1} \cong R/\mathfrak{p}_i$. However, since M is Noetherian this chain must stop, so we obtain a finite filtration:

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M.$$

Then since Ass is subadditive along short exact sequences, an easy induction shows that

$$\operatorname{Ass} M \subseteq \bigcup_{i=1}^{n} \operatorname{Ass}(M_i/M_{i-1}) = \bigcup_{i=1}^{n} \operatorname{Ass}(R/\mathfrak{p}_i) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}.$$

(b) Let $\mathfrak{p} \in \operatorname{Ass} M$. Then $\mathfrak{p} = (0 : z)$ for some $0 \neq z \in M$, so for any $r \in \mathfrak{p}$, we have $rz = 0$.

Conversely, suppose $r \in R$ such that there exists $0 \neq z \in M$ such that $rz = 0$. Then $r \in (0 : z)$. Let $\mathfrak{q}$ be maximal among proper ideals of the form $(0 : y)$, $y \in M$, which contain $(0 : z)$. It is easy to show that $\mathfrak{q}$ is prime, and hence we have $r \in \mathfrak{q} \in \operatorname{Ass} M$.

□

Corollary. *A finitely generated module M over a Noetherian ring R has only finitely many minimal primes.*

Proof. All the minimal primes of M sit in $\operatorname{Ass} M$, which is a finite set. □