Math 614: Lecture notes

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Natural transformations and equivalences of categories[1]

Let $F, G : \mathscr{C} \to \mathscr{D}$ be functors (In this lecture, all functors are covariant.). A *natural transformation* of functors, written $\alpha : F \stackrel{\cdot}{\to} G$, consists of a choice, for every $C \in \operatorname{Ob}\mathscr{C}$, of a morphism $\alpha_C : FC \to GC$ in $\mathscr{D}$ which is "natural in C", in the sense that for any morphism $f : C \to C'$ in $\mathscr{C}$, we have $\alpha_{C'} \circ Ff = Gf \circ \alpha_C$. In other words, the following diagram commutes:

$$\begin{array}{ccccc} C & \quad & F(C) & \xrightarrow{\alpha_C} & G(C) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ C' & & F(C') & \xrightarrow{\alpha_{C'}} & G(C') \end{array}$$

We say that α is a natural transformation from F to G.

If $\alpha : F \stackrel{\cdot}{\to} G$ and $\beta : G \stackrel{\cdot}{\to} H$ are natural transformations, where $F, G, H : \mathscr{C} \to \mathscr{D}$ are functors from $\mathscr{C}$ to $\mathscr{D}$, then we can define the *composition* $\beta \circ \alpha : F \stackrel{\cdot}{\to} H$ via $(\beta \circ \alpha)_C = \beta_C \circ \alpha_C$ for any $C \in \operatorname{Ob}\mathscr{C}$. The composition of any two natural transformations is a natural transformation.

For any functor $F : \mathscr{C} \to \mathscr{D}$, there is a natural transformation $1_F : F \stackrel{\cdot}{\to} F$, called the *identity transformation*, which is defined so that $(1_F)_C = 1_{FC}$ for any $C \in \operatorname{Ob}\mathscr{C}$. If $\alpha : F \stackrel{\cdot}{\to} G$ and $\beta : G \stackrel{\cdot}{\to} F$ are natural transformations such that $\beta \circ \alpha = 1_F$

If $\alpha : F \stackrel{\cdot}{\to} G$ and $\beta : G \stackrel{\cdot}{\to} F$ are natural transformations such that $\beta \circ \alpha = 1_F$ and $\alpha \circ \beta = 1_G$, we say that α is an *isomorphism* from F to G, and we may write $F \cong G$.

[1]These notes may be a little bare-bones, as I don't remember exactly what I covered.

If $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are functors such that $G \circ F = 1_{\mathscr{C}}$ and $F \circ G = 1_{\mathscr{D}}$, then we say the categories $\mathscr{C}$ and $\mathscr{D}$ are *isomorphic*. However, there is a weaker, more useful notion. Namely, if $G \circ F \cong 1_{\mathscr{C}}$ and $F \circ G \cong 1_{\mathscr{D}}$, we say that $\mathscr{C}$ and $\mathscr{D}$ are *equivalent*, and that F is an *equivalence of categories*, and we write $\mathscr{C} \simeq \mathscr{D}$.

When two categories are isomorphic, there is a bijection between their sets of objects. However, two equivalent categories are only equivalent up to isomorphisms of objects, so one might have a situation where $\mathscr{C} \simeq \mathscr{D}$ but $\mathscr{D}$ has many more objects than $\mathscr{C}$. But the isomorphism classes match, and equivalent categories really are 'equivalent' in the ways that matter.

Adjoint functors

Definition. Let $\mathscr{C}$, $\mathscr{D}$ be categories, and $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathscr{C}$ be functors. Suppose that for every pair of objects $C \in \operatorname{Ob} \mathscr{C}, D \in \operatorname{Ob} \mathscr{D}$, there is a bijection $\varphi_{C,D} : \operatorname{Hom}_{\mathscr{D}}(FC, D) \to \operatorname{Hom}_{\mathscr{C}}(C, GD)$, which is "natural in C and D" in the sense that for any $r : C \to C'$ in $\mathscr{C}$, $\alpha : FC' \to D$, and $t : D \to D'$ in $\mathscr{D}$, we have $\varphi_{C,D'}(t \circ \alpha \circ Fr) = Gt \circ \varphi_{C',D}(\alpha) \circ r$. Then we say that $(F, G, \varphi) : \mathscr{C} \to \mathscr{D}$ is an *adjunction*.[2]

The naturality condition can be stated in terms of the commutativity of the following diagram for any such r, α, t, where $_*$ denotes left-composition and * denotes right-composition:

$$
\begin{array}{ccccccc}
 & & \operatorname{Hom}_{\mathscr{D}}(FC', D) & \xrightarrow{\varphi_{C',D}} & \operatorname{Hom}_{\mathscr{C}}(C', GD) & & C' \\
 & & \downarrow{\scriptstyle (Fr)^*} & & \downarrow{\scriptstyle r^*} & & \uparrow{\scriptstyle r} \\
D & & \operatorname{Hom}_{\mathscr{D}}(FC, D) & \xrightarrow{\varphi_{C,D}} & \operatorname{Hom}_{\mathscr{C}}(C, GD) & & C \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t_*} & & \downarrow{\scriptstyle (Gt)_*} & & \\
D' & & \operatorname{Hom}_{\mathscr{D}}(FC, D') & \xrightarrow{\varphi_{C,D'}} & \operatorname{Hom}_{\mathscr{C}}(C, GD') & &
\end{array}
$$

In particular, for any object $C \in \operatorname{Ob} \mathscr{C}$, there is an isomorphism $\varphi_{C,FC} : \operatorname{Hom}_{\mathscr{D}}(FC, FC) \to \operatorname{Hom}_{\mathscr{C}}(C, GFC)$. We set $\eta_C := \varphi_{C,FC}(1_{FC}) : C \to GFC$. These morphisms are particularly important because for any $\alpha : FC \to D$,

[2]Alternate terminology: 'F is left-adjoint to G', 'G is right-adjoint to F', or '$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$ is an adjunction.'

we have $\varphi(\alpha) = \varphi(\alpha \circ 1_{FC}) = G\alpha \circ \varphi(1_{FC}) = G\alpha \circ \eta_C$. So if one knows η_C for every $C \in \operatorname{Ob}\mathscr{C}$ and one knows the action of G on morphisms of $\mathscr{C}$, one can compute every $\varphi_{C,D}$.

In fact, the collection of morphisms η_C defines a natural transformation $\eta : 1_{\mathscr{C}} \dot{\to} GF$. To see this, note that for any morphism $r : C \to C'$ in $\mathscr{C}$, we have $\eta_{C'} \circ r = \varphi(1_{FC'}) \circ r = \varphi(1_{FC'} \circ Fr) = \varphi(Fr) = GFr \circ \eta_C$, which means that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & GFC \\ \Big\downarrow{\scriptstyle r} & & \Big\downarrow{\scriptstyle GFr} \\ C' & \xrightarrow{\eta_{C'}} & GFC' \end{array}$$

which is exactly what is required to define a natural transformation. We call η the *unit* of the adjunction.

Dually, for any object $D \in \operatorname{Ob}\mathscr{D}$, there is an isomorphism $\varphi_{GD,D} : \operatorname{Hom}_{\mathscr{D}}(FGD, D) \to \operatorname{Hom}_{\mathscr{C}}(GD, GD)$. We set $\varepsilon_D := \varphi^{-1}_{GD,D}(1_{GD})$. Then for any morphism $\beta : C \to GD$, we have $\varphi(\varepsilon_D \circ F\beta) = \varphi(\varepsilon_D) \circ \beta = 1_{GD} \circ \beta = \beta$, so that $\varphi^{-1}(\beta) = \varepsilon_D \circ F\beta$. Hence, the action of φ^{-1} is defined by the morphisms ε_D and the action of the functor F on morphisms. Also, it is straightforward to check that the morphisms ε_D give a natural transformation $\varepsilon : FG \dot{\to} 1_{\mathscr{D}}$, called the *counit* of the adjunction.

So an alternate way to define an adjunction is to start by giving a unit and a counit, and define φ and φ^{-1} from there.

Examples

1. Abelianization

   Let $U : \mathbf{Ab} \to \mathbf{Group}$ be the forgetful functor, let $F : \mathbf{Group} \to \mathbf{Ab}$ be the abelianization functor. For any group H, let $\eta_H : H \to UFH = H/[H, H]$ be the quotient map, and for any abelian group A, let $\varepsilon_A : A = UFA \to A$ be the identity map. Then for any $\alpha : FH \to A$, define $\varphi_{H,A}(\alpha) := \alpha \circ \eta_H : H \to UA$. Then $\varphi_{H,A}$ is an isomorphism for every H and A, it satisfies the required naturality properties, and $\varepsilon_A = \varphi^{-1}(1_{UA})$ for every A, as required. Hence, $(F, G, \varphi) : \mathbf{Group} \to \mathbf{Ab}$ is an adjunction.

2. Free object functors

Let $\mathscr{C}$ be a category with a forgetful functor $U : \mathscr{C} \to \mathbf{Set}$. If U has a left-adjoint $F : \mathbf{Set} \to \mathscr{C}$, we call it the *free object functor*, and for any set $X \in \mathscr{C}$, $F(X)$ is called the *free $\mathscr{C}$-object on X*. Then the unit $\eta_X : X \to UFX$ is called *insertion of generators* (typically it will be injective), and the counit $\varepsilon_C : FUC \to C$ is called *free projection* (typically it will be surjective).

For example, if $\mathscr{C} = \mathbf{Ab}$, there is a free object functor $F : \mathbf{Set} \to \mathbf{Ab}$, where $F(X)$ is indeed the free abelian group on the set X. Similarly, there is a functor $F : \mathbf{Set} \to \mathbf{Group}$, whose object function gives the free group on a set, which is indeed left-adjoint to the forgetful functor $U : \mathbf{Group} \to \mathbf{Set}$. We will also see free object functors for the categories $\mathbf{CRng}$, ${}_R\mathbf{Mod}$, and ${}_R\mathbf{Alg}$, all of which are fundamental constructions in commutative algebra

3. Products and coproducts For any category $\mathscr{C}$, consider the diagonal functor $\Delta : \mathscr{C} \to \mathscr{C} \times \mathscr{C}$. If Δ has a left-adjoint $\coprod : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, we call it the *coproduct*, and we say that $\mathscr{C}$ *has binary coproducts*. We write $\coprod(C, D) = C \amalg D$ and $\coprod(f, g) = f \amalg g$. On the other hand, if Δ has a *right* adjoint $\prod : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, we call it the *product* and we say that $\mathscr{C}$ *has binary products*. We write $\prod(C, D) = C \times D$ and $\prod(f, g) = f \times g$. (Similarly, for a fixed set S, we say Δ has S-indexed coproducts or products if $\Delta_S : \mathscr{C} \to \mathscr{C}^S$ has a left adjoint ($\prod_S : \mathscr{C}^S \to \mathscr{C}$) or right adjoint ($\coprod_S : \mathscr{C}^S \to \mathscr{C}$) respectively.) As stated (in the case of product) in an exercise, the product and coproduct defined in this way satisfy the familiar universal properties.