Math 614: Lecture notes

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Extension and contraction of ideals

Let $\varphi : R \to S$ be a ring map, $I \subseteq R$ an ideal of R, and $J \subseteq S$ an ideal of S.

The *extension of* I *to* S, written $\varphi(I)S$, is the ideal of S generated by the set $\varphi(I)$. It is also written IS (by analogue with the case where φ is a ring inclusion), or I^{e} if the ring map is taken as understood. An ideal of S of the form $\varphi(I)S$ is called an *extended* ideal.

The *contraction of* J *to* R is the ideal $\varphi^{-1}(J)$ of R. It is also written $J \cap R$ (by analogue with the case where φ is a ring inclusion), or I^{c} if the ring map is taken as understood. An ideal of R of the form $J \cap R$ is called a *contracted* ideal.

The following properties are easily verified:

1. $I \subseteq I^{\mathrm{ec}}$
2. $J \supseteq J^{\mathrm{ce}}$
3. $J^{\mathrm{c}} = J^{\mathrm{cec}}$
4. $I^{\mathrm{e}} = I^{\mathrm{ece}}$
5. There is an order-preserving bijection between the set of contracted ideals of R and the set of extended ideals of S.

Free modules, free semigroups, semigroup algebras, and free algebras

Fix a ring R. We construct here a left-adjoint $F : \mathbf{Set} \to {}_R\mathbf{Mod}$ to the forgetful functor $U : {}_R\mathbf{Mod} \to \mathbf{Set}$. But first, a definition:

Definition. A subset S of an R-module M is *linearly independent* if whenever $\sum_{i=1}^n r_i s_i = 0$, with $r_i \in R$, $s_i \in S$, and $s_1, \ldots, s_n$ distinct elements of S, it follows that $r_1 = r_2 = \cdots = r_n$.

Let X be a set. As a set, let $F(X) :=$ the set of all X-tuples $(r_x)_{x \in X}$ of elements of R such that $r_x = 0$ for all but finitely many $x \in X$. Alternately, if for each $y \in X$ we let $e_y := (\delta_{y,x})_{x \in X}$, where δ returns 1 for identical elements and 0 for different elements, then $F(X) =$ all formal R-linear combinations $\sum_{i=1}^n r_i e_{x_i}$ of $\{e_x \mid x \in X\}$, where the empty sum is the zero element.

Addition on $F(X)$ is defined componentwise: $(r_x) + (s_x) = (r_x + s_x)$. The R-action is also performed componentwise: $r \cdot (r_x) := (r r_x)$. This defines an R-module structure on $F(X)$.

F is defined on set maps $f : X \to Y$ by setting $f(\sum_{i=1}^n r_i e_{x_i}) := \sum_{i=1}^n r_i e_{f(x_i)}$. Since the set X in the module $F(X)$ is, by construction, linearly independent, this way of defining a map of modules is well-defined.

Finally, for a set X and a module M, we need to define a bijection $\varphi_{X,M} : \mathrm{Hom}_R(FX, M) \to \mathrm{Hom}_{\mathbf{Set}}(X, UM)$ which is natural in X and M. Given $\alpha : FX \to M$, we define $\varphi(\alpha) : X \to UM$ by the rule $\varphi(\alpha)(x) := \alpha(e_x)$. It is clear that φ is a well-defined set map.

φ is injective: If $\varphi(\alpha) = \varphi(\beta)$, then for every $x \in X$, we have $\alpha(e_x) = \beta(e_x)$, so that for any $(r_x) \in FX$, we have $\alpha((r_x)) = \alpha(\sum_{x \in X} r_x e_x) = \sum_{x \in X} r_x \alpha(e_x) = \sum_{x \in X} r_x \beta(e_x) = \beta((r_x))$, and hence $\alpha = \beta$.

φ is surjective: Let $h : X \to UM$ be a set map. Then define the module map $g : FX \to M$ by the rule $g(\sum_x r_x e_x) := \sum_x r_x h(x)$. This is well-defined because the e_x are linearly independent, and we have $\varphi(g)(x) = g(e_x) = h(x)$ for any x, so $\varphi(g) = h$.

Naturality is verified just as easily, and left as a vacuous exercise.

Consequences of the free module functor

Recall that for every adjunction $(\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}, \varphi)$, we get a unit map $\eta_X = \varphi(1_{FX}) : X \to GFX$ for every $X \in \mathrm{Ob}\,\mathscr{D}$, and a counit map $\epsilon_C = \varphi^{-1}(1_{GC}) : FGC \to C$. In the case of the adjunction ${}_R\mathbf{Mod} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Set}$, η_X is the *insertion of generators*, which sends each $x \mapsto e_x$. More interesting, however, is the counit $\epsilon_M : FUM \to M$, which sends, for each $m \in M$, $e_m \mapsto m$. Hence we have that ϵ_M is a *surjection*, which means that *every R-module is a quotient of a free module.*

Of course, ϵ_M is in some sense the least efficient surjection from a free module onto M. We'll come back to this point later, when our objects of choice will be finitely generated modules over a Noetherian ring.

Also, note that the notion of linear independence is essentially equivalent to the notion of a free submodule. Namely, if S is a linearly independent subset of M, then there is an injective map $F(S) \hookrightarrow M$. If, conversely, we have that $F(S) \hookrightarrow M$, then the image of S in M is a linearly independent subset of M.

One can use this fact, along with Zorn's lemma, to show that if R is a field, then every R-module is free (i.e. isomorphic to the free module on some set).

A sequence of adjunctions

Let **CSgp** be the category of (commutative) semigroups with identity. For a ring A, there is a forgetful functor $V : {}_A\mathbf{Alg} \to \mathbf{CSgp}$ such that $V(R)$:=the set R, with its *multiplicative* structure. Also, there is the obvious forgetful functor $U : \mathbf{CSgp} \to \mathbf{Set}$. We will construct a *sequence* of adjunctions:

$$\mathbf{Set} \underset{U}{\overset{S}{\rightleftarrows}} \mathbf{CSgp} \underset{V}{\overset{G}{\rightleftarrows}} {}_A\mathbf{Alg}$$

i.e. (S, U, φ) is an adjunction and (G, V, φ') is an adjunction. It is a routine fact that the *composition* of adjunctions is well-defined, and gives an adjunction, so that in this case we get an adjunction $(G \circ S, U \circ V, \varphi' \circ \varphi) : \mathbf{Set} \to {}_A\mathbf{Alg}$. Then $G \circ S$ is the *free A-algebra functor*, but $A[X] := G(S(X))$ is better known as the *polynomial ring on the set X with coefficients in the ring A*.

Constructions

The *elements* of $S(X)$ are the X-tuples $(n_x)_{x \in X}$, $n_x \in \mathbb{N}$, such that $n_x = 0$ for all but finitely many $x \in X$.

The *multiplication* is given by $(n_x)(p_x) := (n_x + p_x)$, and $1 = (0)_{x \in X}$.

Another way to think of $S(X)$ is by identifying, for each $x \in X$, the $X - tuple$ where the x-position is 1 and every other position is 0. Then the elements are finite products $x_1^{n_1} \dots x_r^{n_r}$ of powers of elements of X, which are made to commute, and the multiplication consists of adding exponents.

Then if $f : X \to Y$ is a set map, $S(f)(\prod_{i=1}^r x_i^{n_i}) := \prod_{i=1}^r f(x_i)^{n_i}$.

The adjunction $\varphi : \mathrm{Hom}_{\mathbf{CSgp}}(FX, T) \to \mathrm{Hom}_{\mathbf{Set}}(X, UT)$ is given by $\varphi(\alpha)(x) = \alpha(x)$. We can copy the above steps to show that S is a functor and φ gives a bijection, natural in X and T.

Now for a semigroup S, we need to construct the "semigroup A-algebra on S": $G(S) :=: A[S]$.

The *elements* of $A[S]$ will be all finite sums $\sum_{i=1}^{n} a_i s_i \mid a_i \in A, s_i \in S$, the s_i are distinct , $n \in \mathbb{N}$. As usual, this can be identified with (US)-tuples $(a_s)_{s \in S}$, $a_s \in A$, where all but finitely many terms are 0.

The *addition* is given componentwise.

The *multiplication* is defined by

$$(\sum_{i=1}^{n} a_i s_i) \cdot (\sum_{i=1}^{n} b_i s_i) = \sum_i \sum_j (a_i b_j) \cdot (s_i s_j)$$

and then collapse terms if necessary.

In particular, $A[x] := A[\{x\}] := A[S(\{x\})] = A[\{1, x, x^2, \dots\}]$ is the usual polynomial ring on one element x. Similarly for any set X, $A[X]$ is the familiar polynomial ring on the set X.

If S is a group, then $A[S]$ is also known as the *group algebra* on S with coefficients in A. By analogy, we always call $A[S]$ the *semigroup algebra* on S with coefficients in A.

As in the module case, for any A-algebra R, the counit of the composite adjunction above gives us a surjection from the polynomial ring on the *set* R onto the ring R. Hence, *every ring is a quotient ring of a polynomial ring.* If R is the image of a polynomial ring on a *finite set* over A, then we say that R is *finitely generated* as an A-algebra. (Note that this is *not* the same as being finitely generated as an A-*module*.)

Direct sums and direct products of modules

Definition. Let X be a set, R a ring, and $\{M_x \mid x \in X\}$ be an X-indexed set of R-modules. The *direct product* $P = \prod_{x \in X} M_x$ is defined as follows: The elements of P are the X-tuples $(m_x)_{x \in X}$, where each $m_x \in M_x$. Sum and R-action are performed componentwise: $(m_x) + (n_x) = (m_x + n_x)$ and $r(m_x) = (rm_x)$.

The *direct sum* $S = \bigoplus_{x \in X} M_x$ is the submodule of P consisting of those X-tuples $(m_x)_{x \in X}$ such that for all but finitely many $x \in X$, $m_x = 0$.

In general, direct sums of modules seem to come up more than direct products, although as can be seen in the definition, they coincide in the case of a finite index set. When the index set is finite, the tendency is to use the symbol $\oplus$ in preference to the symbols $\times$ and Π.

Note that a free module on the set X is exactly $F(X) = \bigoplus_{x \in X} R$. In general, the direct sum of an X-indexed collection of *identical* modules M is called the X-*copower* of M, and written $M^{\oplus X}$, or if X is a finite set and $\#(X) = n$, we write $M^{\oplus n}$. Thus, the free module on n elements is written $R^{\oplus n}$. (Sometimes by abuse of notation it is even written R^n.)