Math 614: Lecture notes

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Localization: Rings and Modules of Fractions

We now tackle the problem of *fractions*. Geometrically, this corresponds to "zeroing in" on a point or subvariety. Algebraically, it corresponds to increasing the set of allowable denominators from R. In both cases, one often can "reduce" a problem to the local case, so that if one can prove a particular statement about all local rings, one then has a proof for all rings.

In general, we will be localizing rings and modules 'at' multiplicative sets:

Definition. Let R be a ring. A *multiplicative set* $W \subseteq R$ is a nonempty subset which is closed under multiplication.

Examples of multiplicative sets:

1. Any ideal of R.
2. For any ideal I of R, the set $1 + I$.
3. If V is a multiplicative set, so are the sets $V \cup \{0\}$ and $V \cup \{1\}$.
4. For any $f \in R$, the set $\{1, f, f^2, f^3, \dots\}$.
5. For any set $X \subseteq \operatorname{Spec} R$ of primes, the set $R \setminus \bigcup X$.

The localized module $W^{-1}M$

Let R be a ring, W a multiplicative set, and M an R-module. Let $P := \{(m, w) \mid m \in M, w \in W\}$. We assign the following equivalence relation to P:

$$(m, w) \sim (m', w') \iff \exists w'' \in W \text{ with } w''w'm = w''wm'.$$

(Think of this equivalence relation as a generalization of the technique of cross-multiplication to determine whether two rational fractions are equal.)

Showing that $\sim$ is an equivalence relation is straightforward. Then set $W^{-1}M := P/\sim$, that is the set P modulo the equivalence relation $\sim$. For $m \in M$ and $w \in W$, write $\frac{m}{w} := m/w :=$ the equivalence class of the element (m, w) in $P/\sim$.

We give $W^{-1}M$ an R-module structure as follows: Addition is given by $\frac{m}{w} + \frac{m'}{w'} := \frac{w'm+wm'}{ww'}$, and the R-action is given by $r(m/w) := (rm)/w$. The zero element is $\frac{0}{w}$ for any $w \in W$. (If $w, w' \in W$, then $w0 = w'0 \Rightarrow \frac{0}{w} = \frac{0}{w'}$.)

In the case $M = R$, we actually have that $S = W^{-1}R$ is a ring. The multiplication is given by $\frac{r}{w}\frac{r'}{w'} := \frac{rr'}{ww'}$. Negation is given by $-\frac{r}{w} := \frac{-r}{w}$. And $1_S := \frac{w}{w}$, where $w \in W$ is arbitrary.

Indeed, the R-module structure of $W^{-1}M$ gives it the structure of a $W^{-1}R$-module: the $W^{-1}R$-action is given by $\frac{r}{w}\frac{m}{w'} := \frac{rm}{ww'}$.

We always have an R-module map $l_W := l_{W,M} : M \to W^{-1}M$ (called the *localization map*), given by $l_W(z) = \frac{wz}{w}$. In the case $M = R$, the map $l_{W,R}$ is a ring homomorphism.

Moreover, $W^{-1} : {}_R\mathbf{Mod} \to {}_{W^{-1}R}\mathbf{Mod}$ is a functor if for an R-module map $f : M \to N$, we define $W^{-1}f : W^{-1}M \to W^{-1}N$ by $(W^{-1}f)(\frac{z}{w}) := \frac{f(z)}{w}$.

Facts about $W^{-1}R$ and $W^{-1}M$

1. If V is a multiplicative set and $U \subseteq R^\times$ (i.e. U consists of units of R), then setting W to be the multiplicative set generated by $V \cup U$, we have $W^{-1}M = V^{-1}M$. In particular if $W = V \cup \{1\}$ we get isomorphic localizations, so we often assume our multiplicative sets *do* contain 1. In this case $l_W(z) = \frac{z}{1}$.

2. $W^{-1}R = 0 \iff 0 \in W$.

   For if $0 \in W$, then for any $r \in R$ and $w \in W$, we have $\frac{r}{w} = \frac{0}{0} \cdot \frac{r}{w} = \frac{0r}{0w} = \frac{0}{0w} = 0$. On the other hand if $W^{-1}R = 0$, then $\frac{w}{w} = \frac{0}{w}$, so that for some $w' \in W$, we have $w'ww = w'w0 = 0$, and $0 = w'w^2 \in W$.

   For this reason, we often assume our multiplicative sets do *not* contain 0.

3. For any R-module M, $M \xrightarrow{l_W} W^{-1}M$ is injective $\iff$ W consists of nonzerodivisors of M.

For if $w \in W$ is a zerodivisor of M, there is some nonzero $z \in M$ with $wz = 0$, in which case $l_W(z) = \frac{wz}{w} = \frac{0}{w} = 0$, so l_W is not injective. Conversely, if W consists of nonzerodivisors and $l_W(z) = 0$, then $\frac{wz}{w} = \frac{0}{w}$, so there is some $w' \in W$ with $w'(w^2 z) = 0$, but $w'w^2 \in W$, which are nonzerodivisors of M, so $z = 0$. Hence, l_W is injective.

4. As a matter of notation, if $W = R \setminus \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec} R$, we write $M_\mathfrak{p} := W^{-1}M$. Note that if $\mathfrak{p} \subseteq \mathfrak{q}$ are two primes, then $(M_\mathfrak{q})_\mathfrak{p} = M_\mathfrak{p}$. If $W = \{1, f, f^2, \dots\}$, we write $M_f := W^{-1}M$. Note that $(M_f)_g = M_{fg}$.

Examples

- $\mathbb{Z}_{(0)} = \mathbb{Q}$, and the construction of this localization is exactly the classical construction of $\mathbb{Q}$ from $\mathbb{Z}$.
- For any integral domain D, $D_{(0)} =: \operatorname{Frac}(D)$ is a field, called the *fraction field of D*.
- For any ring R, let $W :=$ all nonzerodivisors of R. Then for any multiplicative set V such that $l_{V,R}$ is injective, we have a sequence of inclusions $R \hookrightarrow V^{-1}R \hookrightarrow W^{-1}R$. That is, $W^{-1}R$ is the largest localization of R into which R injects. We set $Q(R) := W^{-1}R$ and call it the *total quotient ring of R*. In particular if R is a domain, we have $Q(R) = \operatorname{Frac}(R)$.
- If $p > 0$ is a prime number, $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} \mid (n,p) = 1\}$. This ring is very important in algebraic number theory; its *completion* (a concept we probably won't get to in this course) is the ring of *p-adic integers*.

Localization and operations on submodules

First note that for any $z \in M$, $w \in W$, we have $\frac{z}{w} = 0 \iff$ for some $w' \in W$, we have $w'z = 0$.

Also, localization commutes with most module operations one cares about. First, note that the localization of a submodule is a submodule. That is, if $j : N \hookrightarrow M$ is injective, then so is $W^{-1}j : W^{-1}N \hookrightarrow W^{-1}M$. To see this, suppose $\frac{j(n)}{w} = (W^{-1}j)(\frac{n}{w}) = 0$. Then there is some $w' \in W$ such that $j(w'n) = w'j(n) = 0$. But since j is injective, it follows that $w'n = 0$, so that $\frac{n}{w} = 0$. For this reason, if N is a submodule of M, we can identify $W^{-1}N$ as a submodule of $W^{-1}M$.

With this identification, we have the following properties for submodules N, P of M:

1. $W^{-1}(N+P) = W^{-1}N + W^{-1}P$.
2. $W^{-1}(N \cap P) = W^{-1}N \cap W^{-1}P$.
3. $W^{-1}(M/N) \cong (W^{-1}M)/(W^{-1}N)$.

   The first of these properties is trivial to verify. The second follows from collecting common denominators. As for the third, if $p : M \to M/N$ is the canonical surjection, let $\pi := W^{-1}p : W^{-1}M \to W^{-1}(M/N)$. Then it's easy to see that π is a surjection, and that its kernel is exactly $W^{-1}N$.

4. If M is finitely generated[1], $W \cap \operatorname{Ann} M \neq \emptyset \iff W^{-1}M = 0$.
5. If M is finitely generated, then $W^{-1}\operatorname{Ann}(M) = \operatorname{Ann}(W^{-1}M)$.
6. If P is finitely generated, then $W^{-1}(N :_R P) = (W^{-1}N :_{W^{-1}R} W^{-1}P)$.

   To prove (5), first note that the forward containment always holds: If x annihilates M, then $\frac{x}{w}$ annihilates $W^{-1}M$. Conversely, suppose $M = \sum_{i=1}^{n} Rz_i$ is finitely generated and $\frac{x}{w} \in \operatorname{Ann} W^{-1}M$. Then $\frac{xz_i}{w} = 0$ for all i, which means that for each i, there is some $w_i \in W$ with $w_i x z_i = 0$. Let $w' = \prod_{i=1}^{n} w_i$. Then $w' x z_i = 0$ for all i, so that $w'x \in \operatorname{Ann} M$, and so $\frac{x}{w} \in W^{-1}\operatorname{Ann} M$.

   To prove (4), again the forward containment holds for any M, and for finitely generated M we have $W^{-1}M = 0 \iff \operatorname{Ann}(W^{-1}M) = W^{-1}R \iff$ (by (5)) $W^{-1}\operatorname{Ann} M = W^{-1}R \iff \frac{x}{w} = 1$ for some $x \in \operatorname{Ann} M$, $w \in W \iff w'w = w'x$ for some $w' \in W$, and then this is an element of $W \cap \operatorname{Ann} M$.

   As for (6), recall that $N : P = \operatorname{Ann} \frac{N+P}{N}$, and then apply (5) and (1).

[1] As was pointed out by a student, this does *not* hold if M is not finitely generated. In fact, both (4) and (5) typically fail. For example, let R be an integral domain which is not a field, $Q = R_{(0)}$ its field of fractions, $W = R \setminus \{0\}$, and $M = Q/R$ as an R-module. Then $W^{-1}M = 0$, so that $\operatorname{Ann} W^{-1}M = W^{-1}R = Q$, but $\operatorname{Ann} M = 0$, so that $W \cap \operatorname{Ann} M = \emptyset$ and $W^{-1}\operatorname{Ann} M = 0$.

Ideals of R *vs.* ideals of $W^{-1}R$

Fix the localization map $i = l_W : R \to W^{-1}R$. For an ideal $I \subseteq R$, $I^{\mathrm{e}} := i(I)W^{-1}R$ and for an ideal $J \subseteq W^{-1}R$, $J^{\mathrm{c}} := i^{-1}(J)$. Then

Proposition. *1. Every ideal of $W^{-1}R$ is extended from R.*

2. Let $\bar{W}$ be the image of W in R/I. Then $\bar{W}^{-1}(R/I) = (W^{-1}R)/I^{\mathrm{e}}$.

3. $I^{\mathrm{ec}} = \bigcup_{w \in W}(I : w)$.

4. $I^{\mathrm{e}} = W^{-1}R \iff I \cap W \neq \emptyset$.

5. I is contracted $\iff$ W consists of nonzerodivisors of R/I.

Proof. 1. Let J be an ideal of $W^{-1}R$, and $\frac{x}{w} \in J$. Then $i(x) = \frac{x}{1} = w \cdot \frac{x}{w} \in J$, so $x \in J^{\mathrm{c}}$, hence $\frac{1}{w}i(x) = \frac{x}{w} \in J^{\mathrm{ce}}$. Thus, $J = J^{\mathrm{ce}}$.

2. By (3) from the module section, we need only show that the R-module $W^{-1}I$ is an ideal of $W^{-1}R$. This follows because for $i \in I$, $\frac{x}{w}\frac{i}{w'} = \frac{xi}{ww'} \in W^{-1}I$.

3. If $wx \in I$, then $\frac{x}{w'} \in I^{\mathrm{e}}$ for any w', and hence $x \in I^{\mathrm{ec}}$. Conversely if $x \in I^{\mathrm{ec}}$, then $\frac{x}{1} \in I^{\mathrm{e}}$ which means that $\frac{x}{1} = \frac{x'}{w}$ for some $x' \in I$, so that we have $w'wx = w'x' \in I$.

4. Use (2), pass to $\bar{R} = R/I$, and use the fact that $I \cap W \neq \emptyset \iff \bar{0} \in \bar{W}$.

5. By (3), $I^{\mathrm{ec}} = I \iff (I : w) = I$ for all $w \in W \iff (0 :_{R/I} w) = 0$ in R/I.

□

For a multiplicative set $W \subseteq R$, let $C(W) := \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap W = \emptyset\}$. Then a good exercise consists of showing that there is a homeomorphism between $\operatorname{Spec} W^{-1}R$ and $C(W)$.

Corollary. *If $\mathcal{N}$ stands for nilradical, then $\mathcal{N}(W^{-1}R) = W^{-1}\mathcal{N}(R)$.*

Corollary. *If $X \subseteq \operatorname{Spec} R$ and $W = R \setminus \bigcup X$, then $\operatorname{Spec}(W^{-1}R) \cong \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in X\}$*

In the exercises you will show that *all* localizations can be described this way.

Corollary. *$\kappa(\mathfrak{p}) := (R/\mathfrak{p})_{(0)} = \overline{(R \setminus \mathfrak{p})}^{-1}(R/\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is a field (called the* residue field of R at $\mathfrak{p}$*).*