Math 614: Lecture notes

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As a first use of localization, we will prove the following proposition, that 'vanishing of a module is a local property.'

Proposition. *Let R be a ring and M an R-module. Then the following are equivalent:*

1. $M = 0$.
2. $W^{-1}M = 0$ *for every multiplicative set* $W \subseteq R$.
3. $M_{\mathfrak{p}} = 0$ *for every* $\mathfrak{p} \in \operatorname{Spec} R$.
4. $M_{\mathfrak{m}} = 0$ *for every* $\mathfrak{m} \in \Omega(R)$.

Proof. $1 \Rightarrow 2$ because $\frac{0}{w} = 0$. $2 \Rightarrow 3 \Rightarrow 4$ are trivial. All that remains is to show that $4 \Rightarrow 1$.

Suppose $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$, and take any $z \in M$. Then for any $\mathfrak{m}$, we have $M_{\mathfrak{m}} = 0$, and in particular $\frac{z}{1} = 0$ in $M_{\mathfrak{m}}$. This means that there is some $w \in R \setminus \mathfrak{m}$ such that $wz = 0$, and so $w \in (0 : z)$, and $(0 : z) \nsubseteq \mathfrak{m}$. So $(0 : z)$ is contained in no maximal ideal of R, and hence must be the unit ideal. Thus, $0 = 1_R \cdot z = z$. $\square$

Sequences, complexes, and exactness

Definition. A *sequence* of R-modules is defined to be a set of R-modules, which can be indexed by some interval in $\mathbb{Z}$, along with a linear sequence of module homomorphisms connecting them. For example:

$$M_0, \quad M_1 \xrightarrow{d_1} M_0, \quad \cdots \xrightarrow{d_3} M_2 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0,$$

$M_0 \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} M_{-2} \xrightarrow{d_{-2}} \cdots$, and $\cdots \xrightarrow{d_3} M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \cdots$

are all sequences. Any of the above sequences, with the numbers going down as the arrows go to the right, and $d_j : M_j \to M_{j-1}$ for each d_j, we may label $(M., d.)$ for short.

A sequence $(M., d.)$ is a *complex* if $d_j \circ d_{j+1} = 0$ whenever the composition is defined. In other words, $\operatorname{Im} d_{j+1} \subseteq \operatorname{Ker} d_j$.

A complex $(M., d.)$ is *exact at* M_j if either (1) at least one of d_j or d_{j+1} is not defined, or (2) $\operatorname{Im} d_{j+1} = \operatorname{Ker} d_j$. A complex is *exact* if it is exact at M_j for all j where M_j is defined.

Special kinds of exact sequences and functors

The primary example of an exact sequence is the following: Let N be an R-submodule of the R-module M. Let $j : N \to M$ be the inclusion map, and $\pi : M \to M/N$ the natural projection. Then the sequence

$$0 \to N \xrightarrow{j} M \xrightarrow{\pi} M/N \to 0$$

is exact.

In general, a *short exact sequence* is a 5-term exact complex with zeros at either ends – i.e. it has the form

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0.$$

Note that any short-exact sequence is isomorphic to the sequence one gets from a module-inclusion. Namely, the sequence above is isomorphic to the sequence $0 \to i(M') \to M \to M/i(M') \to 0$.

A *left-exact sequence* is an exact sequence of the form $0 \to M' \to M \to M''$.

A *right-exact sequence* is an exact sequence of the form $M' \to M \to M'' \to 0$.

Definition. Let R, T be rings. A functor $F : {}_R\mathbf{Mod} \to {}_T\mathbf{Mod}$ is *additive* if for any R-modules M, N and homomorphisms $f, g \in \operatorname{Hom}_R(M, N)$, we have $F(f + g) = F(f) + F(g)$ and $F(-f) = -F(f)$.

Note that if $(M., d.)$ is a complex and F is an additive functor, then the sequence $(F(M.), F(d.))$ is also a complex, since we have $F(d_j) \circ F(d_{j+1}) = F(d_j \circ d_{j+1}) = F(0) = 0$.

We say that an additive functor F is *left-exact* if it takes left-exact sequences to left-exact sequences, *right-exact* if it takes right-exact sequences to right-exact sequences, and *exact* if it takes short exact sequences to short exact sequences[1].

In the previous lecture, we have shown that if $j : N \hookrightarrow M$ is an inclusion of modules, then $W^{-1}j$ is injective and $W^{-1}(M/N) = W^{-1}M/W^{-1}N$. In the terminology of this lecture, this says that *the functor* $W^{-1} : {}_R\mathbf{Mod} \to {}_{W^{-1}R}\mathbf{Mod}$ *is exact.*

You will show in the exercises that if $(F, G, \varphi) : {}_R\mathbf{Mod} \to {}_T\mathbf{Mod}$ is an adjunction, where F and G are additive, then F is right-exact and G is left-exact.

Hom

Let A, B be rings which are *not necessarily commutative.* Then we have the notions of a *left* A-module (that is, A acts on the left, so we get an action $z \mapsto az$, along with the rule $(aa')z = a(a'z)$) with module-category ${}_A\mathbf{Mod}$, a *right* B-module (where B acts on the right, so we get an action $z \mapsto zb$, along with the rule $z(bb') = (zb)b'$) with module category $\mathbf{Mod}_B$, and an A-B-*bimodule* (that is, M is both a left A-module and a right B-module, and the actions are compatible in the sense that $a(zb) = (az)b$ for all $a \in A$, $b \in B$, $z \in M$, so we may write azb unambiguously) with module category ${}_A\mathbf{Mod}_B$.

Note that ${}_A\mathbf{Mod} \cong {}_A\mathbf{Mod}_{\mathbb{Z}}$ and $\mathbf{Mod}_B \cong {}_{\mathbb{Z}}\mathbf{Mod}_B$ are equivalences of categories.

If $M, N \in {}_A\mathbf{Mod}$, then the left A-module homomorphisms $H = \mathrm{Hom}_A(M, N)$ from M to N form an Abelian group. In general there is no natural left or right A-action on this group. However, if $M \in {}_A\mathbf{Mod}_B$ and $N \in {}_A\mathbf{Mod}_C$, then $\mathrm{Hom}_A(M, N) \in {}_B\mathbf{Mod}_C$, with bimodule action given by $(bfc)(m) := f(mb)\cdot c$. Indeed, we have a functor $\mathrm{Hom}_A : {}_A\mathbf{Mod}_B{}^{\mathrm{op}} \times {}_A\mathbf{Mod}_C \to {}_B\mathbf{Mod}_C$.

Now, fix $N \in {}_A\mathbf{Mod}B$. Then there is a functor $H = \mathrm{Hom}_A(N, -) : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$, where the left B-module structure on any $\mathrm{Hom}_A(N, P)$ comes from the right B-module structure of N. We will show that H has a left adjoint $T = N \otimes_B - : {}_B\mathbf{Mod} \to {}_A\mathbf{Mod}$.

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[1] This is equivalent to saying that F applied to *any* exact sequence gives an exact sequence, but we will not need the fact that these are equivalent conditions.

Tensor product

For a left B-module M, we define the left A-module $N \otimes_B M$ as follows:

Let $N \times M$ be the set-theoretic product of N and P, and $F := F(N, M) :=$ the free Abelian group on $N \times M$. Let G be the subgroup of F generated by all elements of each of the following three forms, for all $m, m' \in M$, $n, n' \in N$, and $b \in B$:

$$(n, m+m') - (n, m) - (n, m'), \quad (n+n', m) - (n, m) - (n', m), \quad (nb, m) - (n, bm)$$

Then set $N \otimes_B M := F/G$, and denote the equivalence classes of (n, m) with the tensor symbol, as in $n \otimes m := (n, m) + G \in F/G$. This notation is ambiguous, but universally used.

The group $N \otimes_B M$ has a left A-module structure given by $a(n \otimes m) := (an) \otimes m$. To see that this is well-defined, we define it first on F (that is, $a(n, m) = (an, m)$), and we see that the action of any a on an element of G gives another element of G. Hence, we get an A-action on $N \otimes_B M = F/G$.

Tensor is left-adjoint to Hom

We must define a natural isomorphism $\varphi_{M,P} : \mathrm{Hom}_A(N \otimes_B M, P) \to \mathrm{Hom}_B(M, \mathrm{Hom}_A(N, P))$. That is, for any $g : N \otimes_B M \to P$, we must give a B-linear map $\varphi(g) : M \to \mathrm{Hom}_A(N, P)$, so that in particular, for every $m \in M$, $\varphi(g)(m) : N \to P$ is A-linear.

We define φ by $\varphi(g)(m)(n) := g(n \otimes m)$. One must verify the following facts:

1. For every $g : N \otimes_B M \to P$ and $m \in M$, $\varphi(g)(m)$ is A-linear.
2. Each $\varphi(g)$ is B-linear.
3. φ is a group homomorphism[2].
4. φ is injective.
5. φ is surjective.
6. $\varphi_{M,P}$ is natural in M and P

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[2] Actually we don't need to prove this, but it's useful to know

We leave out the verifications of parts 1-3 and 6. For part 4, suppose $\varphi(g) = 0$. Then $g(n \otimes m) = 0$ for every $n \in N$ and $m \in M$. But every element of $N \otimes_B M$ is of the form $\sum_{i=1}^k n_i \otimes m_i$, so we have $g(\sum_i n_i \otimes m_i) = \sum_i g(n_i \otimes m_i) = \sum_i 0 = 0$. Hence, $g = 0$.

For part 5, let $h : M \to \mathrm{Hom}_A(N, P)$ be B-linear, and "define" $g : N \otimes_B M \to P$ by $g(n \otimes m) := h(m)(n)$. To show that this is well-defined, define it on basis elements of F as $g'(n, m) = h(m)(n)$ and show that g' vanishes on G, as follows: $g'(n + n', m) = h(m)(n + n') = h(m)(n) + h(m)(n') = g'(n, m) + g'(n', m)$, similarly for $(n, m + m')$, and $g'(nb, m) = h(m)(nb) := (b \cdot h(m))(n) = h(bm)(n) = g'(n, bm)$. Hence, g is well-defined as the induced map from g'. Then we have $\varphi(g)(m)(n) = g(n \otimes m) = h(m)(n)$, so that $\varphi(g) = h$.

Corollary. *For an A-B-bimodule N, the functor* $\mathrm{Hom}_A(N, -) : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ *is left-exact, and the functor* $N \otimes_B - : {}_B\mathbf{Mod} \to {}_A\mathbf{Mod}$ *is right-exact.*

Proof. By the above, the tensor functor is left-adjoint to the hom functor. Thus by the exercise, they have the required exactness properties. □

Note that the functors $A \otimes_A - : {}_A\mathbf{Mod} \to {}_A\mathbf{Mod}$ and $\mathrm{Hom}_A(A, -) : {}_A\mathbf{Mod} \to {}_A\mathbf{Mod}$ are both isomorphic to the identity functor on ${}_A\mathbf{Mod}$. For an A-module M, the isomorphism $M \to \mathrm{Hom}_A(A, M)$ is given by $m \mapsto (a \mapsto am)$, and the isomorphism $A \otimes M \to M$ is given by $\sum_{i=1}^k a_i \otimes m_i \mapsto \sum_{i=1}^k a_i m_i$.

Extension, restriction, and 'evaluation' of scalars

Now we return to the commutative ring setting. Let $g : R \to S$ be a ring homomorphism, and N an S-module. Then there is a functor $\rho : {}_S\mathbf{Mod} \to {}_R\mathbf{Mod}$, called *restriction of scalars*, which sends N to N, but with R-action given by $r \cdot z := g(r)z$, and it sends every S-homomorphism of modules to itself.

If one considers S to be an S-R-bimodule via *right* action by g, then ρ is isomorphic to the functor $\mathrm{Hom}_S(S, -)$. And we have

$$\mathrm{Hom}_S(S \otimes_R M, N) \xrightarrow{\varphi} \mathrm{Hom}_R(M, \mathrm{Hom}_S(S, N)) \cong \mathrm{Hom}_R(M, \rho(N)),$$

so that $S \otimes_R - : {}_R\mathbf{Mod} \to {}_S\mathbf{Mod}$ (a functor we call *extension of scalars*) is left-adjoint to ρ.

On the other hand, if one considers S to be an R-S-bimodule via *left* action by g, then ρ is isomorphic to the functor $S \otimes_S -$. And we have

$$\operatorname{Hom}_R(\rho(N), M) \cong \operatorname{Hom}_R(S \otimes_S N, M) \xrightarrow{\varphi} \operatorname{Hom}_S(N, \operatorname{Hom}_R(S, M))$$

so that $\operatorname{Hom}_R(S, -) : {}_S\mathbf{Mod} \to {}_R\mathbf{Mod}$ is right-adjoint to ρ. (I don't know a commonly used name for this functor; for now I will call it *evaluation of scalars*.)

Hence, evaluation of scalars is left-exact, extension of scalars is right-exact, and restriction of scalars is exact.