Math 614: Lecture notes

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More on Hom, $\bigoplus$, $\bigotimes$, and Exactness

We showed that for an R-module N, the functor $N \otimes_R - : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$ is left-adjoint to $\mathrm{Hom}_R(N, -) : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$. You will show in an exercise that any left-adjoint functor is right exact and any right-adjoint is left exact. Hence $\mathbf{N} \otimes_{\mathbf{R}} -$ **is right-exact** and $\mathrm{Hom}_{\mathbf{R}}(\mathbf{N}, -)$ **is left-exact**. It turns out that the contravariant functor $\mathrm{Hom}_R(-, N) : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}^{op}$ is also 'left-exact', in the following sense:

Proposition. *Applying* $\mathrm{Hom}_R(-, N)$ *to any right-exact sequence of* R*-modules yields a left-exact sequence.*

Proof. Take any right-exact sequence

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$

Applying $\mathrm{Hom}_R(-, N)$ to it gives the sequence

$$0 \to \mathrm{Hom}_R(M'', N) \xrightarrow{g^*} \mathrm{Hom}_R(M, N) \xrightarrow{f^*} \mathrm{Hom}_R(M', N)$$

where the notation means that for composable maps u and v, $u^*(v) := v \circ u$. We need to show that g^* is injective, that $f^* \circ g^* = 0$, and that $\mathrm{Ker}\, f^* \subseteq \mathrm{Im}\, g^*$.

Let $h : M'' \to N$ be in $\mathrm{Ker}\, g^*$, which means that $h \circ g = g^*(h) = 0$. Take any $z \in M''$. Then since g is surjective, there is some $y \in M$ with $g(y) = z$. Then $h(z) = h(g(y)) = (g^*(h))(y) = 0$. Thus, $h = 0$, so g^* is injective.

For any $h : M'' \to N$, we have $(f^* \circ g^*)(h) = (f^*)(h \circ g) = (h \circ g) \circ f = h \circ (g \circ f) = h \circ 0 = 0$. Hence, $f^* \circ g^* = 0$.

Finally, let $h : M \to N$ with $h \in \operatorname{Ker} f^*$. This means that $h \circ f = 0$. Then "define" a map $\tilde{h} : M'' \to N$ as follows. For any $z \in M''$, there is some $y \in M$ with $g(y) = z$, and we set $\tilde{h}(z) := h(y)$. This is well-defined because if $z = g(y) = g(y')$, then $y - y' \in \operatorname{Ker} g = \operatorname{Im} f \subseteq \operatorname{Ker} h$, which means that $h(y') = h(y)$. Moreover, $h = \tilde{h} \circ g = g^*(\tilde{h}) \in \operatorname{Im} g^*$. $\square$

You may ask: "For which N are these functors exact?" This is an important question, and it leads to the following set of definitions:

Definition. If the functor $N \otimes_R -$ is exact (or equivalently, if it preserves injections), we say that N is *flat*. If $\operatorname{Hom}_R(N, -)$ is exact (equiv: it preserves surjections), we say that N is *projective*. If $\operatorname{Hom}_R(-, N)$ is exact (equiv: it sends injections to surjections), we say that N is *injective*.

"Most" R-modules are neither flat, projective, nor injective, even if $R = \mathbb{Z}$ or $R = k[x]$ (k a field, x an indeterminate), as the following examples show:

1. Let $R := \mathbb{Z}$, $N := \mathbb{Z}/2\mathbb{Z}$, and consider the following short exact sequence

$$\mathcal{S} : 0 \to 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \to 0$$

   (a) *N is not projective.*

   Indeed, $p_* := \operatorname{Hom}_R(N, p) : \operatorname{Hom}(\mathbb{Z}/(2), \mathbb{Z}) \to \operatorname{Hom}(\mathbb{Z}/(2), \mathbb{Z}/(2))$ is not surjective:

   We have that $\operatorname{Hom}_R(\mathbb{Z}/(2), \mathbb{Z}) = 0$, since if $g : \mathbb{Z}/(2) \to \mathbb{Z}$ is a homomorphism, then $2g(\bar{1}) = g(\bar{2}) = g(\bar{(0)}) = 0$, and since 2 is a non-zerodivisor in $\mathbb{Z}$, we have $g(\bar{1}) = 0$. Hence, $g = 0$, so that $\operatorname{Hom}_R(\mathbb{Z}/(2), \mathbb{Z}) = 0$.

   On the other hand, the identity map on $\mathbb{Z}/(2)$ is a nonzero element in $\operatorname{Hom}(\mathbb{Z}/(2), \mathbb{Z}/(2))$. Thus, p_* maps a zero module to a nonzero module, so it cannot be a surjection.

   (b) *N is not flat.*

   Indeed, $i' := 1_N \otimes_R i : (\mathbb{Z}/(2)) \otimes 2\mathbb{Z} \to (\mathbb{Z}/(2)) \otimes \mathbb{Z}$ is not injective: The element $\bar{1} \otimes 2 \in (\mathbb{Z}/(2)) \otimes 2\mathbb{Z}$ is nonzero, but $i'(\bar{1} \otimes 2) = \bar{1} \otimes (2 \cdot 1) = \overline{1 \cdot 2} \otimes 1 = \bar{2} \otimes 1 = \bar{0} \otimes 1 = 0$ in $(\mathbb{Z}/(2)) \otimes \mathbb{Z}$. (Indeed, upon further analysis one can see that i' is the zero map!).

(c) *N is not injective.*

Indeed, $i^* := \mathrm{Hom}_R(i, N) : \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/(2)) \to \mathrm{Hom}(2\mathbb{Z}, \mathbb{Z}/(2))$ is not surjective:

Define $g : 2\mathbb{Z} \to \mathbb{Z}/(2)$ by the rule $g(2n) = \bar{n} =$ (the parity of n). If $g = i^*(h)$ for some $h : \mathbb{Z} \to \mathbb{Z}/(2)$, then $g(2) = (i^*(h))(2) = h(i(2)) = h(2) = 2h(1) = \bar{2}h(1) = \bar{0}$, but $g(2) = \bar{1} \neq 0$, so there can be no such h. In other words, g is not in the image of i^*.

2. Let $R := k[x]$, where k is any field (or any commutative ring!) and x is an indeterminate over k. Let $N = k[x]/(x) \simeq k$.[1] Consider the short exact sequence of R-modules:

$$\mathcal{S}' : 0 \to xR \xrightarrow{i} R \xrightarrow{p} N \to 0.$$

This example has the same structure as example 1, as follows:

(a) *N is not projective.*

$p_* := \mathrm{Hom}_R(N, p)$ is not surjective, because $\mathrm{Hom}_R(N, R) = 0$ but $\mathrm{Hom}_R(N, N) \cong k \neq 0$.

(b) *N is not flat.*

$i' := 1_N \otimes_R i$ is not injective, because $\bar{1} \otimes x \in N \otimes_R xR$ is nonzero, but its i'-image in $N \otimes_R R$ is zero.

(c) *N is not injective.*

We have that $i^* := \mathrm{Hom}_R(i, N)$ is not surjective, because defining $g : xR \to k$ by the rule $g(xp) := \bar{p} =$(the constant term of p) is a well-defined map which is not the i^*-image of any element in $\mathrm{Hom}_R(R, k)$.

3. Indeed, the above argument goes through whenever R is a ring that contains an element f which is neither a unit nor a zerodivisor. Then we let $N = R/fR$, and use the short exact sequence

$$\mathcal{S} : 0 \to fR \xrightarrow{i} R \xrightarrow{p} R/fR \to 0.$$

Then $\mathrm{Hom}_R(N, p)$ is not surjective, $1_N \otimes_R i$ is not injective, and $\mathrm{Hom}_R(i, N)$ is not surjective. In example 1, $f = 2$, and in example 2, $f = x$.

[1] I denoted it in the quotient module format to indicate the structure of N as an R-module; namely, for any polynomial $p = p(x) \in R$, there is a unique $c \in k$ and $q = q(x) \in R$ such that $p = c + xq$. Then for $d \in N = k$, we set $p \cdot d := cd$.

Properties of tensor products

- Note that the fact that R is commutative means that we always have a natural isomorphism $M \otimes_R N \cong N \otimes_R M$, which sends $m \otimes n$ to $n \otimes m$ (and back again) (i.e. tensor products of modules over a commutative ring are *commutative*).

- Regardless of commutativity of the rings involved, if $M \in \mathbf{Mod}_R$, $N \in {}_R\mathbf{Mod}_S$, and $P \in {}_S\mathbf{Mod}$, we have a canonical isomorphism $(M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P)$, which sends $(m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$ (i.e. tensor product is *associative*).

- **Tensor products commute with direct sums**

  Recall that if X is a set and $\{M_x \mid x \in X\}$ is an X-indexed set of R-modules, then $M := \bigoplus_{x \in X} M_x$ is the set of X-tuples $(m_x)_{x \in X}$, where each $m_x \in M_x$, with the property that $m_x = 0$ for all but finitely many $x \in X$. For each $y \in X$ and each $m \in M_y$, let the symbol me_y stand for the element (m_x) of M such that $m_x = 0$ if $x \neq y$ and $m_y = m$. One must be careful here because e_y is *not* an element of M. However, with this notation, every element of M can be expressed uniquely in the form $\sum_{i=1}^n m_i e_{x_i}$, where $\{x_1, \ldots, x_n\}$ is a set of n distinct elements of X and $0 \neq m_i \in M_{x_i}$. (As usual, the empty sum is the zero element.)

  In particular, if $M_x = M$ for all $x \in X$, we write $M^{\oplus X} := \bigoplus_{x \in X} M$, and the free module on the set X is $R^{\oplus X} := R^X := \bigoplus_{x \in X} R$.

  For $y \in X$, let $i_y : M_y \to M$ be defined by $i_y(m) := me_y$. Then it is easy to show that for any R-module N and any X-indexed set of maps $g_x : M_x \to N$, there is a unique map $g : M \to N$ such that $g \circ i_x = g_x$ for all $x \in X$. Namely, let $g(me_x) := g_x(m)$ for any $x \in X$, $m \in M_x$, and extend linearly. Moreover, given any $(M', \{j_x : M_x \to M'\}_{x \in X})$ that have this universal property, we have $M \cong M'$.

  Proposition. *Let $M = \bigoplus_{x \in X} M_x$, where the M_x are R-modules, and let N be an R-module. Then*

  $$N \otimes_R M \cong \bigoplus_{x \in X} (N \otimes_R M_x).$$

  Hence,

  $$(\bigoplus_{y \in Y} N_y) \otimes_R (\bigoplus_{x \in X} M_x) \cong \bigoplus_{(x,y) \in X \times Y} N_y \otimes_R M_x,$$

and in particular, we have

$$R^{\oplus Y} \otimes_R R^{\oplus X} \cong R^{\oplus (X \times Y)}.$$

That is, the tensor product of a free module on X with a free module on Y is a free module on $X \times Y$.

Proof. For the first statement, we need only show that $N \otimes_R M$ satisfies the universal property of direct sums. Let $j_x := (1_N \otimes_R i_x) : N \otimes_R M_x \to N \otimes_R M$, and let $\{g_x : N \otimes_R M_x \to P\}_{x \in X}$ be an X-indexed set of R-module maps. Then if $\varphi : \operatorname{Hom}(N \otimes_R M_x, P) \to \operatorname{Hom}(M_x, \operatorname{Hom}(N, P))$ is the Hom $-\otimes$ adjunction, we have a collection $\varphi(g_x) : M_x \to \operatorname{Hom}(N, P)$, so that there is a unique $h : M \to \operatorname{Hom}(N, P)$ such that $h \circ i_x = \varphi(g_x)$ for all $x \in X$. Applying φ^{-1}, we have

$$g_x = \varphi^{-1}(\varphi(g_x)) = \varphi^{-1}(h \circ i_x) = \varphi^{-1}(h) \circ (1_N \otimes i_x),$$

so that $g = \varphi^{-1}(h) : N \otimes_R M \to P$ is the desired map. To see uniqueness, suppose $u : N \otimes_R M \to P$ has the property that $u \circ (1_N \otimes i_x) = g_x$ for all x. Then applying φ, we have $\varphi(g_x) = \varphi(u) \circ i_x$ for all x, and then the uniqueness of h gives that $h = \varphi(u)$, so that $g = \varphi^{-1}(h) = \varphi^{-1}(\varphi(u)) = u$.

The second and third statements follow from the canonical isomorphism $N \otimes_R M \cong M \otimes_R N$. □

Actually, the above proof can be altered slightly to show that *left adjoints preserve coproducts*. (Dually, right-adjoints preserve products.)

- If $N = \operatorname{Cok} g$, then $M \otimes_R N \cong \operatorname{Cok}(1_M \otimes g)$.

  Proof. Right-exactness of $M \otimes_R -$. □

- If I is an ideal and M an R-module, then $M \otimes_R (R/I) \cong M/IM$.

  Proof. Since $R/I = \operatorname{Cok}(I \hookrightarrow R)$, we have $M \otimes_R (R/I) \cong \operatorname{Cok}(1_M \otimes (I \hookrightarrow R)) \cong (M \otimes R)/\operatorname{Im}((M \otimes I) \to M \otimes R) \cong M/IM$ □