Math 614: Lecture notes

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More on Hom, $\bigoplus$, $\bigotimes$, and Exactness, cont'd

Proposition 1. *If $i : M' \hookrightarrow M$ and $j : N' \hookrightarrow N$ are R-module inclusions, then*

$$\frac{M}{M'} \otimes_R \frac{N}{N'} \cong \frac{M \otimes_R N}{\operatorname{Im}(i \otimes_R 1_N) + \operatorname{Im}(1_M \otimes_R j)}$$

Proof. We have short exact sequences

$$\mathcal{S} : 0 \to M' \xrightarrow{i} M \xrightarrow{\pi} M'' \to 0$$

and

$$\mathcal{T} : 0 \to N' \xrightarrow{i} N \xrightarrow{\pi} N'' \to 0$$

and the right-exactness of tensor product means we have the following commutative diagram, where the rows and column are exact:

$$\begin{tikzcd}
& M \otimes_R N' \arrow[r, "b"] \arrow[d, "d"] & M'' \otimes_R N' \arrow[r] \arrow[d, "e"] & 0 \\
M' \otimes_R N \arrow[r, "f"] & M \otimes_R N \arrow[r, "g"] & M'' \otimes_R N \arrow[r] \arrow[d, "h"] & 0 \\
& & M'' \otimes_R N'' \arrow[r] \arrow[d] & 0 \\
& & 0 &
\end{tikzcd}$$

By the exactness of the column, $M'' \otimes_R N'' \cong \frac{M'' \otimes_R N}{\operatorname{Im} e}$. Since g is surjective, $M'' \otimes_R N = \operatorname{Im} g$, so that $M'' \otimes_R N'' \cong \frac{\operatorname{Im} g}{\operatorname{Im} e}$. (In particular, $\operatorname{Im} e \subseteq \operatorname{Im} g$, but this already follows from the surjectivity of b and g). The exactness of the middle row means that g induces an isomorphism $\tilde{g} : \frac{M \otimes_R N}{\operatorname{Im} f} \xrightarrow{\sim} \operatorname{Im} g$.

Moreover, $g^{-1}(\operatorname{Im} e) = \operatorname{Im} d + \operatorname{Im} f$. To see this, first take some $z \in g^{-1}(\operatorname{Im} e)$. Then by surjectivity of b, there is some x with $g(z) = e(b(x)) = g(d(x))$, so $z - d(x) \in \operatorname{Ker} g = \operatorname{Im} f$, whence $z \in \operatorname{Im} d + \operatorname{Im} f$. Conversely, take some element $d(x) + f(y) \in \operatorname{Im} d + \operatorname{Im} f$. Then $g(d(x) + f(y)) = g(d(x)) = e(b(x)) \in \operatorname{Im} e$, so that $d(x) + f(y) \in g^{-1}(\operatorname{Im} e)$.

Thus, under the isomorphism $\tilde{g}$, we have $\tilde{g}^{-1}(\operatorname{Im} e) = (\operatorname{Im} d + \operatorname{Im} f)/\operatorname{Im} f$. Hence, we have

$$M'' \otimes_R N'' \cong \frac{M'' \otimes_R N}{\operatorname{Im} e} \cong \frac{\operatorname{Im} g}{\operatorname{Im} e} \cong \frac{(M \otimes_R N)/\operatorname{Im} f}{(\operatorname{Im} d + \operatorname{Im} f)/\operatorname{Im} f} \cong \frac{M \otimes_R N}{\operatorname{Im} d + \operatorname{Im} f}.$$

(The third isomorphism above is induced by $\tilde{g}^{-1}$.) But $d = 1_M \otimes_R j$ and $f = i \otimes_R 1_N$, so we are done. $\square$

The right-exactness of tensor product will allow us to see what *extension of scalars* does to an R-module, at least in terms of its *presentation*. We detour here to discuss this important topic:

Presentations

Let M be any R-module. Then there is an R-linear surjection from a free module onto R. Say X is a set and $R^{\oplus X}$ is a free module that admits an R-linear surjection $\pi : R^{\oplus X} \twoheadrightarrow M$. This map has some kernel, say K, which also admits an R-linear surjection from a free module onto it; say $\pi' : R^{\oplus Y} \twoheadrightarrow K$. Let $j : K \hookrightarrow R^{\oplus X}$ be the inclusion map, and $g = j \circ \pi' : R^{\oplus Y} \to R^{\oplus X}$ the composition. Then it follows easily that the following sequence is (right)-exact:

$$R^{\oplus Y} \xrightarrow{g} R^{\oplus X} \xrightarrow{\pi} M \to 0. \tag{1}$$

A right-exact sequence where the first two modules are free (i.e. a sequence in the form of (1)) is called a *presentation* of M. We have shown that every R-module has a presentation. If X can be chosen to be a finite set, we already know this is equivalent to M being *finitely generated*. If both X and Y can be chosen to be finite sets, we say that M is *finitely presented*.

Note that any map of finitely generated free modules with fixed bases (say $g : R^n \to R^m$) can be represented uniquely by an $m \times n$ matrix of elements of R, in the following sense: Let $e_1, \dots, e_n$ be the fixed basis of R^n and $e'_1, \dots, e'_m$

be the fixed basis of R^m. If g is such a map, let r_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq m)$ be such that for each e_j, $g(e_j) = \sum_{i=1}^{m} r_{ij} e'_i$. Then the $m \times n$ matrix $A = (r_{ij})$ determines the action of g. Moreover, if we represent elements of R^n and R^m by column vectors, where in R^n, $(a_1, \ldots, a_n)^{\mathrm{tr}} := \sum_{j=1}^{n} a_j e_j$, and similarly for R^m, then the usual matrix multiplication by A is exactly the action of g, since

$$A \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \sum\limits_{j=1}^{n} r_{1j} a_j \\ \vdots \\ \sum\limits_{j=1}^{n} r_{mj} a_j \end{pmatrix} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} r_{ij} a_j \right) e'_i = \sum_{j=1}^{n} a_j \left(\sum_{i=1}^{m} r_{ij} e'_i \right)$$

$$= \sum_{j=1}^{n} a_j g(e_j) = g\left(\sum_{j=1}^{n} a_j e_j \right) = g \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Conversely, given an $m \times n$ matrix $A = (r_{ij})$ of elements of R and free modules R^n, R^m with fixed bases $\{e_j\}$ and $\{e'_i\}$ respectively, there is a unique R-linear homomorphism $g : R^n \to R^m$ such that the action of g corresponds to the multiplication of column vectors by A, in terms of the fixed bases.

In the case of free modules that may not be finitely generated, the preceding discussion can be carried through with no change except that each "column" of the matrix can have only finitely many nonzero entries. That is, given a free module map $g : R^{\oplus Y} \to R^{\oplus X}$, where the free modules have fixed bases $\{e_y\}_{y \in Y}$ and $\{e'_x\}_{x \in X}$ respectively, we have $g(e_y) = \sum_{x \in X} r_{xy} e'_x$, which must be a finite sum, and so for each y, only finitely many of the r_{xy} can be nonzero. As above, matrix multiplication by $A = (r_{xy})$ from column vectors in $R^{\oplus Y}$ (which themselves have only finitely many nonzero entries, of course) to column vectors in $R^{\oplus X}$ (which also have the finiteness condition on nonzero entries) corresponds exactly to the action of g, and such matrix multiplication only makes sense anyway if each column of the matrix has only finitely many nonzero entries.

Hence, we can consider any R-module to be the cokernel of the action of such a matrix. In particular, if M is finitely presented, then it is the cokernel of multiplication by a finite matrix of elements of R. This point of view is very useful.

Extension of scalars, applied to presentations

Let $\varphi : R \to S$ be a ring homomorphism, let M be an R-module. We have a presentation as in (1) above, which is, of course, right exact. Moreover, we know that tensor product commutes with direct sum, and that $S \otimes_R R \cong S$ by a natural isomorphism. Hence, we have the following commutative diagram, with exact rows because extension of scalars is right-exact:

$$\begin{tikzcd}
S \otimes_R R^{\oplus Y} \arrow[r, "1\otimes g"] \arrow[d, "u", "(\cong)"'] & S \otimes_R R^{\oplus X} \arrow[r, "1\otimes\pi"] \arrow[d, "v", "(\cong)"'] & S \otimes_R M \arrow[r] \arrow[d, equal] & 0 \\
S^{\oplus Y} \arrow[r, "g'"] & S^{\oplus X} \arrow[r, "\pi'"] & S \otimes_R M \arrow[r] & 0
\end{tikzcd}$$

So to represent $S \otimes_R M$ as a cokernel, it suffices to understand what the action of g' is. Fix bases $\{e_y\}$ for $R^{\oplus Y}$ and $\{e'_x\}$ for $R^{\oplus X}$, and let's say that the R-matrix of representing g in terms of these bases is $(r_{xy})_{x\in X, y\in Y}$. We have $v(1 \otimes re'_x) = \varphi(r)e'_x$ and $u(1 \otimes re_y) = \varphi(r)e_y$ for all $x \in X$, $y \in Y$, and $r \in R$. Thus, for any $y \in Y$, we have

$$\begin{aligned}
g'(e_y) &= v((1 \otimes g)(u^{-1}(e_y))) = v((1 \otimes g)(1 \otimes e_y)) = v(1 \otimes g(e_y)) \\
&= v\left(1 \otimes \sum_{x\in X} r_{xy}e'_x\right) = \sum_{x\in X} v(1 \otimes r_{xy}e'_x) = \sum_{x\in X} \varphi(r_{xy})e'_x
\end{aligned}$$

Hence, the matrix representing g' is exactly $(\varphi(r_{xy}))_{x\in X, y\in Y}$, which means that $S \otimes_R M$ is the cokernel of this matrix.

Thus, the representation of M as the cokernel of an R-matrix leads directly to the representation of $S \otimes_R M$ as the cokernel of an S-matrix, simply by applying φ to each of the entries!

Flat and surjective base change on tensor products

Let $\varphi : R \to S$ be a ring homomorphism and let M, N be S-modules. Then in general, $M \otimes_R N$ has *two different S-module structures!* Let $\alpha(M, N) = M \otimes_R N$ with the S-module structure inherited from M, and $\beta(M, N) = M \otimes_R N$ with the S-module structure inherited from N. In symbols, the S-action on $\alpha(M, N)$ is given by $s(m \otimes n) := (sm) \otimes n$, and the S-action on $\beta(M, N)$ is given by $s(m \otimes n) := m \otimes (sn)$.

To see that this gives two different structures in general, let R be any ring, let$S = R[x]$, where x is an indeterminate over R, and let $\varphi : R \to R[x]$ be the obvious ring inclusion. Let $M = N = S$. Then it is clear that $x \otimes 1 \neq 1 \otimes x$ in $S \otimes_R S$. However, in $\alpha(S, S)$, we have $x \cdot (1 \otimes 1) = x \otimes 1$, and in $\beta(S, S)$, we have $x \cdot (1 \otimes 1) = 1 \otimes x$, so the S-module structures are distinct.

From another point of view, for any two S-modules M and N, one can form the tensor products $M \otimes_S N$ and $M \otimes_R N$. In general there is a surjection $\sigma_{M,N} : M \otimes_R N \twoheadrightarrow M \otimes_S N$, since the defining relations on the tensor product over R form a subset of the defining relations on the tensor product over S. [In particular $(rm, n) = (\varphi(r)m, n)$ and $(m, rn) = (m, \varphi(r)n)$.] It happens that $\sigma_{M,N}$ is an isomorphism iff $\alpha(M, N) = \beta(M, N)$.

However, for some ring homomorphisms $\varphi : R \to S$, we have $\alpha(M, N) = \beta(M, N)$ (i.e. $\sigma_{M,N}$ is injective) for all pairs M, N of S-modules.

One such example is where $S = R/I$ and φ is the canonical surjection. Then for any S-module M and $r \in R$, we have for any $z \in M$ that $rz = \bar{r}z$. So for any $s = \bar{r} \in S$ ($r \in R$), $m \in M$, and $n \in N$, we have in $M \otimes_R N$ that

$$sm \otimes n = \bar{r}m \otimes n = rm \otimes n = m \otimes rn = m \otimes \bar{r}n = m \otimes sn,$$

which shows that $\alpha(M, N) = \beta(M, N)$.

Another example is when $S = W^{-1}R$, and $\varphi = l_W$ is the localization map. For any $r \in R$, $w \in W$, $m \in M$, and $n \in N$, we have

$$m\frac{r}{w} \otimes n = \frac{m}{w} \otimes rn = \frac{m}{w} \otimes \frac{wrn}{w} = \frac{m}{w} \otimes w\frac{rn}{w} = \frac{m}{w}w \otimes \frac{rn}{w} = m \otimes \frac{r}{w}n.$$

Proposition 2. *Let $\varphi : R \to S$ be a ring homomorphism with the property that $\sigma_{P,Q}$ is an isomorphism for all pairs P, Q of S-modules. Then for any pair M, N of R-modules, we have*

$$S \otimes_R (M \otimes_R N) \cong (S \otimes_R M) \otimes_S (S \otimes_R N).$$

In other words, tensor product commutes with such a base change.

Proof.

$$\begin{aligned} S \otimes_R (M \otimes_R N) &\cong (S \otimes_S S) \otimes_R (M \otimes_R N) \\ &\cong (S \otimes_R S) \otimes_R (M \otimes_R N) && \text{(via } \sigma_{S,S}) \\ &\cong (S \otimes_R M) \otimes_R (S \otimes_R N) && \text{(by associativity of tensor)} \\ &\cong (S \otimes_R M) \otimes_S (S \otimes_R N) && \text{(via } \sigma_{S \otimes_R M, S \otimes_R N}) \end{aligned}$$

□

Corollary. *For any ring R, multiplicative set $W \subseteq R$, any ideal $I \subseteq R$, and any pair M, N of R-modules, we have*

$$(R/I) \otimes_R (M \otimes_R N) \cong (M/IM) \otimes_{R/I} N/IN$$

and

$$W^{-1}(M \otimes_R N) \cong W^{-1}M \otimes_{W^{-1}R} W^{-1}N.$$

Proof. By Proposition 2 and the discussion immediately above it. □