Math 614: Lecture notes

Sep 26, 2007

Left-adjoint functors commute with coproducts

Definition. Given a category $\mathscr{C}$ and a set X, we can define a category $\mathscr{C}^X$ as follows: The objects of $\mathscr{C}^X$ are X-indexed sets of objects $(C_x)_{x\in X}$ of $\mathscr{C}$. A morphism $(C_x) \to (C'_x)$ is given by an X-indexed set $(f_x : C_x \to C'_x)_{x\in X}$ of $\mathscr{C}$-morphisms. Composition is done coordinatewise: $(g_x) \circ (f_x) := (g_x \circ f_x)$. For any functor $F : \mathscr{C} \to \mathscr{D}$, define the functor $F^X : \mathscr{C}^X \to \mathscr{D}^X$ coordinatewise on both objects and morphisms. That is, $F^X((C_x)_{x\in X}) := (F(C_x))_{x\in X}$ and $F^X((g_x)_{x\in X}) := (F(g_x)_{x\in X})$.

We define the diagonal functor $\Delta := \Delta_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}^X$ by sending $C \mapsto (C)_{x\in X}$ and $f \mapsto (f)_{x\in X}$. (i.e. It sends each object and morphism to "X copies of itself".) If Δ has a left-adjoint $\coprod := \coprod_{\mathscr{C}} : \mathscr{C}^X \to \mathscr{C}$, we call it a *coproduct*, and we say that *$\mathscr{C}$ has X-indexed coproducts.* (Dually for the right-adjoint to Δ, which is called a product $\prod$.)

I said vaguely in a previous lecture that left-adjoint functors commute with coproducts. This is now made precise in the following series of lemmas, the first three of which are given without proof:

Lemma 1. *Let $\mathscr{C}, \mathscr{D}, \mathscr{E}$ be categories, and let $(F, G, \varphi) : \mathscr{C} \to \mathscr{D}$ and $(F', G', \varphi') : \mathscr{D} \to \mathscr{E}$ be adjunctions. Then $(F' \circ F, G \circ G', \varphi \circ \varphi') : \mathscr{C} \to \mathscr{E}$ is also an adjunction.*

Proof. Purely formal. □

Lemma 2. *Let $\mathscr{C}, \mathscr{D}$ be categories, let $(F, G, \varphi) : \mathscr{C} \to \mathscr{D}$ be an adjunction, and let X be any set. Then $(F^X, G^X, \varphi^X) : \mathscr{C}^X \to \mathscr{D}^X$ is also an adjunction.*

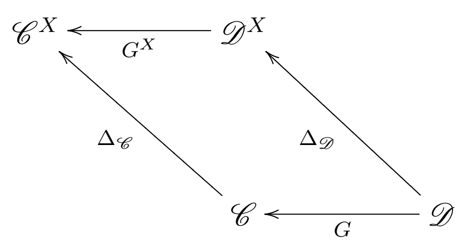
Proof. Formal. □

Lemma 3. *Let $F, F' : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors such that both F and F' are left-adjoint to G. Then there is an isomorphism of functors $F \cong F'$*

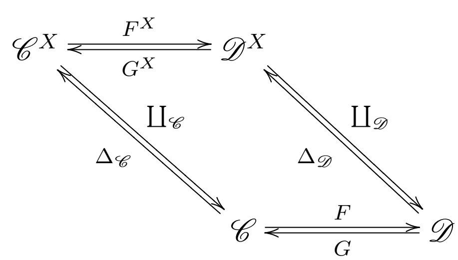
Proof. Tricky, but formal. One needs to construct the isomorphism from the available data. $\square$

Lemma 4. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Suppose that both $\mathscr{C}$ and $\mathscr{D}$ have X-indexed coproducts, and let $\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$ be an adjunction. Then $F \circ \coprod_{\mathscr{C}} \cong \coprod_{\mathscr{D}} \circ F^X : \mathscr{C}^X \to \mathscr{D}$. In particular, $F\left(\coprod_{x \in X} C_x\right) \cong \coprod_{x \in X} F\left(C_x\right)$.*

Proof. Consider the following diagram:



It's easy to see that the diagram commutes. Now let's fill in the diagram with the left-adjoints for each of the maps:



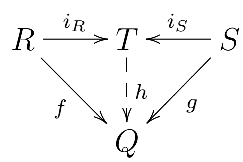
Here F^X is left-adjoint to G^X by Lemma 2. By Lemma 1, we have that $\coprod_{\mathscr{D}} \circ F^X$ is left-adjoint to $G^X \circ \Delta_{\mathscr{D}}$, and also $F \circ \coprod_{\mathscr{C}}$ is left-adjoint to $\Delta_{\mathscr{C}} \circ G$. But $G^X \circ \Delta_{\mathscr{D}} = \Delta_{\mathscr{C}} \circ G$. Hence by Lemma 3, we get the required isomorphism of functors. $\square$

Tensor products of A-algebras as coproducts

Let R, S be A-algebras. The structure maps give, in particular, that R, S are A-modules, so we may form the tensor product $T := R \otimes_A S$. In fact, *this is a coproduct in the category of A-algebras!*

To see this, we must first give an A-algebra structure on T. We let $(r \otimes s) \cdot (r' \otimes s') := rr' \otimes ss'$, and use the distributive law to define the product of arbitrary pairs of T-elements. It is straightforward to check that this gives T the structure of a commutative ring, with $1_T = 1 \otimes 1$ being the multiplicative identity. The map $A \to T$ given by $a \mapsto a \otimes 1 (= 1 \otimes a)$ makes T into an A-algebra, and from now on we consider this to be the structure map of T. Note that there are A-algebra homomorphisms $i_R : R \to T$ and $i_S : S \to T$ given by $i_R(r) := r \otimes 1$ and $i_S(s) := 1 \otimes s$. Moreover, the triple (T, i_R, i_S) has the following *universal property*:

Given any pair $f : R \to Q$ and $g : S \to Q$ of A-algebra homomorphisms, there is a unique A-algebra homomorphism $h : T \to Q$ that makes the following diagram commute:



By the dual to the exercise on products from the first exercise set, it follows that $R \otimes_A S$ *is the coproduct of* R *and* S *in the category of* A*-algebras.*

This fact means that if we know "generators and relations" for the A-algebras R and S, we can get generators and relations for $R \otimes_A S$, as follows:

We have that R is a quotient of a polynomial ring, say $\pi : A[X] \twoheadrightarrow R$, where X is some set. This map has some kernel K_R, which is an ideal of $A[X]$, with inclusion map $i : K_R \hookrightarrow A[X]$. Similarly, we have $p : A[Y] \twoheadrightarrow S$, for some set Y, with some kernel K_S which is an ideal of $A[Y]$, and inclusion map $j : K_S \hookrightarrow A[Y]$.

By Lemma 4, and since disjoint union $\uplus$ is the coproduct operation in **Set**, we have $A[X] \otimes_A A[Y] \cong A[X \uplus Y]$. In other words, the tensor product of the polynomial rings on two sets X and Y is a polynomial ring on their disjoint union. (Another way to say this: The set $\{x \otimes 1 \mid x \in X\} \cup \{1 \otimes y \mid y \in Y\}$ of elements of $A[X] \otimes_A A[Y]$ is algebraically independent.)

We may as well assume that X and Y are disjoint sets, so that $A[X \uplus Y] = A[X, Y]$.

By our work on Proposition 1 of the Sep 24 lecture notes, we have

$$\begin{aligned} R \otimes_A S &\cong \frac{A[X]}{K_R} \otimes_A \frac{A[Y]}{K_S} \cong \frac{A[X] \otimes_A A[Y]}{\operatorname{Im}(i \otimes 1) + \operatorname{Im}(1 \otimes j)} \\ &\cong \frac{A[X,Y]}{K_R A[X,Y] + K_S A[X,Y]} \end{aligned}$$

Free, projective, and flat

Recall that an R-module P is *projective* if $\operatorname{Hom}_R(P, -)$ preserves surjections.

Proposition 1. *The following are equivalent for an R-module P.*

a. P is projective.

b. For any surjection $\pi : M \twoheadrightarrow N$ of R-modules and any map $g : P \to N$, there is a "lifting $\tilde{g} : P \to M$ of g to M". (This just means that $\pi \circ \tilde{g} = g$.)

c. P is a direct summand of a free module.

d. P is a direct summand of any free module that maps onto it.

Proof. (a $\iff$ b): This is just by definition. $\pi_* := \operatorname{Hom}(P, \pi) : \operatorname{Hom}(P, M) \to \operatorname{Hom}(P, N)$ is surjective iff for any $g \in \operatorname{Hom}(P, N)$, there is some $\tilde{g} \in \operatorname{Hom}(P, M)$ with $\pi \circ \tilde{g} = \pi_*(\tilde{g}) = g$.

(c $\Rightarrow$ b): First we show that *free modules are projective.* Let $F = F(X) = R^{\oplus X} = \bigoplus_{x \in X} Re_x$ be a free module on X with basis $\{e_x\}_{x \in X}$. Then let $\pi : M \twoheadrightarrow N$ be a surjection and $h : F \to N$ be any homomorphism. For each $x \in X$, pick some $m_x \in M$ such that $\pi(m_x) = h(e_x)$. Then define $\tilde{h} : F \to M$ on basis elements by $\tilde{h}(e_x) := m_x$. This extends R-linearly to a well-defined homomorphism, and clearly $h = \pi \circ \tilde{h}$.

Now let F be a free module that has P as a direct summand. By the exercises, this is equivalent to the existence of a pair $F \underset{j}{\overset{p}{\rightleftarrows}} P$ of maps such that $p \circ j = 1_P$. Let $M \twoheadrightarrow N$ be a surjection and $g : P \to N$ be a homomorphism. Then $g \circ p : F \to N$ has a lifting $h : F \to M$, which means that $\pi \circ h = g \circ p$. Then $g = g \circ p \circ j = \pi \circ h \circ j$, so that $h \circ j : P \to M$ is a lifting of g.

(d $\Rightarrow$ c) is trivial.

(a $\Rightarrow$ d): Let $\pi : F \to P$ be a surjection, where F is a free module. Since P is projective, the map $\pi_* : \operatorname{Hom}(P, F) \to \operatorname{Hom}(P, P)$ is a surjection. Hence there is some $j \in \operatorname{Hom}_R(P, F)$ such that $\pi \circ j = \pi_*(j) = 1_P$, which by the exercises is equivalent to P being a direct summand of F. $\square$

Proposition 2. *If P is a projective R-module, then P is flat.*

Proof. First we show that *free modules are flat.* Let $F = R^{\oplus X} = \bigoplus_{x \in X} Re_x$ be a free module. Let $i : N \hookrightarrow M$ be an injection of R-modules, and consider the map $F \otimes_R N \xrightarrow{1 \otimes i} F \otimes_R M$. A typical element of $F \otimes_R N$ looks like $z = \sum_{j=1}^{t} e_{x_j} \otimes n_j$, $n_j \in N$, $x_j \in X$. If $\sum_{j=1}^{t} e_{x_j} \otimes i(n_j) = (1 \otimes i)(z) = 0$, then $i(n_j) = 0$ for each j, so that $n_j = 0$ for all j, which certainly implies that $z = 0$. Hence $1 \otimes i$ is injective and F is flat.

By Proposition 1, P is a direct summand of a free module F, so we have the pair of maps $F \underset{j}{\overset{p}{\rightleftarrows}} P$ with $p \circ j = 1_P$. For an injection $i : N \hookrightarrow M$ of R-modules, we want to show that $1_P \otimes_R i$ is injective. We get the following commutative diagram, where (by the exercises) the columns compose to the identity maps:

$$\begin{array}{ccc} P \otimes_R N & \xrightarrow{i'} & P \otimes_R M \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} \\ F \otimes_R N & \xhookrightarrow{i''} & F \otimes_R M \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ P \otimes_R N & \xrightarrow{i'} & P \otimes_R M \end{array}$$

If $i'(x) = 0$, then $j_2(i'(x)) = i''(j_1(x)) = 0$, so that i'' surjective implies that $j_1(x) = 0$, so that we have $x = p_1(j_1(x)) = p_1(0) = 0$. Hence i' is injective, but $i' = 1_P \otimes_R i$, so P is flat. $\square$

Thus, we have shown:

$$\text{Free} \Longrightarrow \text{Projective} \Longrightarrow \text{Flat}.$$

Later we will show that for finitely generated modules over a local ring, all three properties are equivalent.