

Math 614: Lecture notes (Outline)

Sep 28, 2007

Proposition 1 (Tensor Evaluation). *Let A, B be rings, $L \in {}_A\mathbf{Mod}$, $M \in {}_A\mathbf{Mod}_B$, and $N \in {}_B\mathbf{Mod}$. Then there is a natural homomorphism*

$$t_{LMN} : \mathrm{Hom}_A(L, M) \otimes_B N \rightarrow \mathrm{Hom}_A(L, M \otimes_B N).$$

It is an isomorphism if L is finitely presented over A and N is a flat B -module.

Proof outline. The homomorphism is defined by $t_{LMN}(f \otimes n)(l) := f(l) \otimes n$. One must show that this is well-defined, a homomorphism, and natural in L , M , and N .

Also, if $L = A^r$ is a finitely generated free A -module, then t_{LMN} is an isomorphism (natural in M and N), as it is the composition of the following sequence of isomorphisms:

$$\mathrm{Hom}_A(A^r, M) \otimes_B N \cong M^r \otimes_B N \cong (M \otimes_B N)^r \cong \mathrm{Hom}_A(A^r, M \otimes_B N).$$

Now suppose that L is finitely presented, so that we have a (right-)exact sequence of the form $A^s \rightarrow A^r \rightarrow L \rightarrow 0$. Since $\mathrm{Hom}_A(-, M \otimes_B N)$ converts right-exact sequences into left-exact ones, we have the left-exact sequence

$$0 \rightarrow \mathrm{Hom}_A(L, M \otimes_B N) \rightarrow \mathrm{Hom}_A(A^r, M \otimes_B N) \rightarrow \mathrm{Hom}_A(A^s, M \otimes_B N).$$

On the other hand, since $\mathrm{Hom}_A(M, -)$ converts right-exact sequences into left-exact sequences, and the flatness of N means that $(- \otimes_B N)$ preserves left-exact sequences, we have that $\mathrm{Hom}_A(M, -) \otimes_B N$ converts right-exact sequences into left-exact ones, so we get the left-exact sequence

$$0 \rightarrow \mathrm{Hom}_A(L, M) \otimes_B N \rightarrow \mathrm{Hom}_A(A^r, M) \otimes_B N \rightarrow \mathrm{Hom}_A(A^s, M) \otimes_B N.$$

Since all the maps involved are natural, as is t , we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_A(L, M) \otimes_B N & \longrightarrow & \mathrm{Hom}_A(A^r, M) \otimes_B N & \longrightarrow & \mathrm{Hom}_A(A^s, M) \otimes_B N \\
& & \downarrow t_{LMN} & & \cong \downarrow t_{(A^r)MN} & & \cong \downarrow t_{(A^s)MN} \\
0 & \longrightarrow & \mathrm{Hom}_A(L, M \otimes_B N) & \longrightarrow & \mathrm{Hom}_A(A^r, M \otimes_B N) & \longrightarrow & \mathrm{Hom}_A(A^s, M \otimes_B N)
\end{array}$$

By an easy diagram chase, it follows that t_{LMN} is also an isomorphism. \square

Corollary (Hom commutes with flat base change). *Let R be a commutative ring, let L, M be R -modules, and $\varphi : R \rightarrow S$ a ring homomorphism. Then there is an S -linear homomorphism $S \otimes_R \mathrm{Hom}_R(M, N) \rightarrow \mathrm{Hom}_S(S \otimes_R L, S \otimes_R M)$, which is an isomorphism if S is flat over R and L is finitely presented.*

Proof. We have $t_{LMS} : S \otimes_R \mathrm{Hom}_R(L, M) \rightarrow \mathrm{Hom}_R(L, S \otimes_R M)$, which is an isomorphism if S is flat and L is finitely presented. But

$$\mathrm{Hom}_R(L, S \otimes_R M) \cong \mathrm{Hom}_R(L, \mathrm{Hom}_S(S, S \otimes_R M)) \cong \mathrm{Hom}_S(S \otimes_R L, S \otimes_R M)$$

because $S \otimes_R -$ is left-adjoint to $\mathrm{Hom}_S(S, -)$. \square

In particular, if L is finitely presented and $W \subseteq R$ is a multiplicative set, then

$$W^{-1} \mathrm{Hom}_R(L, M) \cong \mathrm{Hom}_{W^{-1}R}(W^{-1}L, W^{-1}M),$$

so that for any prime $\mathfrak{p} \in \mathrm{Spec} R$, we have

$$(\mathrm{Hom}_R(L, M))_{\mathfrak{p}} \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(L_{\mathfrak{p}}, M_{\mathfrak{p}}).$$

Warning: This fails in general if L is not finitely presented. For example, let $R = \mathbb{Z}$, $L = S = \mathbb{Q}$, and $M = \mathbb{Z}$, and $\mathfrak{p} = (0)$, then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$, so that $(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}))_{(0)} = 0$, but $\mathrm{Hom}_{\mathbb{Z}_{(0)}}(\mathbb{Q}_{(0)}, \mathbb{Z}_{(0)}) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q} \neq 0$.

Ass and Supp

Definition. Let M be an R -module. The *support* of M is

$$\mathrm{Supp} M := \{\mathfrak{p} \in \mathrm{Spec} R \mid M_{\mathfrak{p}} \neq 0\}.$$

The *Assassinator* of M is

$$\text{Ass } M := \text{Spec } R \cap \{\text{Ann}_R z \mid z \in M\}.$$

If $\mathfrak{p} \in \text{Ass } M$, we say that \mathfrak{p} is an *associated prime* of M , or \mathfrak{p} is *associated to* M .

Proposition 2. *Let M be an R -module.*

1. $\text{Supp}(M) \subseteq V(\text{Ann } M)$. *If M is finitely generated, then equality holds.*
2. *If $\mathfrak{q} \supseteq \mathfrak{p} \in \text{Supp } M$, then $\mathfrak{q} \in \text{Supp } M$.*
3. $\mathfrak{p} \in \text{Ass } M \iff$ *There is an injective map $j : R/\mathfrak{p} \hookrightarrow M$ of R -modules.*
4. $\text{Ass } M \subseteq \text{Supp } M$.
5. *If $I = (\mathfrak{p} : x)$ for some $x \in R$, then $I = \mathfrak{p}$ or $I = R$.*
6. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence. Then*

$$\text{Supp } M = \text{Supp } L \cup \text{Supp } N \quad \text{and} \quad \text{Ass } M \subseteq \text{Ass } L \cup \text{Ass } N.$$

7. $\text{Supp}(L \otimes_R M) \subseteq \text{Supp } L \cup \text{Supp } M$.

Proof. 1. If $\mathfrak{p} \in \text{Spec } R \setminus V(\text{Ann } M)$, then there is some $x \in \text{Ann } M \setminus \mathfrak{p}$, so for any $\frac{z}{w} \in M_{\mathfrak{p}}$, we have $\frac{z}{w} = \frac{xz}{xw} = \frac{0}{xw} = 0$, so that $\mathfrak{p} \notin \text{Supp } M$.

Conversely, suppose M is finitely generated (by elements z_1, \dots, z_n) and $\mathfrak{p} \notin \text{Supp } M$. Then $M_{\mathfrak{p}} = 0$, so for each i there is some $w_i \in R \setminus \mathfrak{p}$ such that $w_i z_i = 0$. Then $w = \prod_i w_i \in R \setminus \mathfrak{p}$ and $w z_i = 0$ for each i , so that $w \in \text{Ann } M \setminus \mathfrak{p}$, and $\mathfrak{p} \notin V(\text{Ann } M)$.

2. We have $M_{\mathfrak{p}} = (M_{\mathfrak{q}})_{\mathfrak{p}}$, so that if $M_{\mathfrak{p}} \neq 0$, we have $M_{\mathfrak{q}} \neq 0$.
3. If $\mathfrak{p} \in \text{Ass } M$, then $\mathfrak{p} = (0 : z)$ for some $z \in M$. Define $j : R/\mathfrak{p} \rightarrow M$ by $j(\bar{r}) := rz$. This is well-defined because $\mathfrak{p} \subseteq (0 : z)$, and injective because $\mathfrak{p} \supseteq (0 : z)$.

Conversely, if an injective map $j : R/\mathfrak{p} \hookrightarrow M$ exists, let $z := j(\bar{1})$. Then $\mathfrak{p}z = \mathfrak{p}j(\bar{1}) = j(\bar{\mathfrak{p}}) = j(\bar{0}) = 0$, and if $r \notin \mathfrak{p}$ then $rz = rj(\bar{1}) = j(\bar{r}) \neq 0$ by injectivity of j . Hence, $\mathfrak{p} = (0 : z)$.

4. Let $\mathfrak{p} \in \text{Ass } M$. Then there is an exact sequence $0 \rightarrow R/\mathfrak{p} \rightarrow M$, so by exactness of localization we get an exact sequence $0 \rightarrow (R/\mathfrak{p})_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}}$. But $(R/\mathfrak{p})_{\mathfrak{p}}$ is the fraction field of the integral domain R/\mathfrak{p} and so is nonzero, and hence $M_{\mathfrak{p}} \neq 0$ so that $\mathfrak{p} \in \text{Supp } M$.
5. We always have $\mathfrak{p} \subseteq I$, and $I = R \iff x \in \mathfrak{p}$. Now suppose $x \notin \mathfrak{p}$ and let $a \in I$. Then since $ax \in \mathfrak{p}$, it follows that $a \in \mathfrak{p}$. Hence $I = \mathfrak{p}$.
6. It's an elementary fact that the middle term of a short exact sequence is zero if and only if both of the outside terms are zero. Thus, since localization is exact, we have $M_{\mathfrak{p}} = 0 \iff L_{\mathfrak{p}} = N_{\mathfrak{p}} = 0$, from which follows the equation relating to supports.

Now suppose that $\mathfrak{p} \in \text{Ass } M$. Say $\mathfrak{p} = (0 : z)$ for some $z \in M$. Let $Q := Rz \cap L$. If $Q \neq 0$, let $0 \neq rz \in Q$. Then $(0 : rz) = (\mathfrak{p} : r) = \mathfrak{p}$ since $r \notin \mathfrak{p}$, and hence $\mathfrak{p} \in \text{Ass } L$. On the other hand if $Q = 0$, let $I := (0 : \bar{z})$, where \bar{z} is the image of z in N . We have $\bar{z} \neq 0$, so I is a proper ideal, and clearly $\mathfrak{p} \subseteq I$ since \mathfrak{p} annihilates $z \in M$. Suppose $a \in I$. Then $\bar{a}z = 0$, so that $az \in Rz \cap L = 0$, whence $a \in (0 : z) = \mathfrak{p}$. Thus, $I = \mathfrak{p} \in \text{Ass } N$.

7. If $\mathfrak{p} \in \text{Supp}(L \otimes_R M)$, then we have $0 \neq (L \otimes_R M)_{\mathfrak{p}} \cong L_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$, so that $L_{\mathfrak{p}} \neq 0$ and $M_{\mathfrak{p}} \neq 0$.

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