Math 614: Lecture notes

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The topic for today is *category theory.* Categories are fundamental for understanding mathematics, and we will use several concepts from it crucially in this course, especially that of the adjoint functor.

Definition 1. A *category* $\mathscr{C}$ consists of the following data:

- A class $\mathrm{Ob}(\mathscr{C})$, called the *objects* of $\mathscr{C}$.
- For each pair $X, Y \in \mathrm{Ob}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ (also denoted $\mathrm{Mor}(X, Y)$ or $\mathrm{Hom}(X, Y)$ if $\mathscr{C}$ is understood), called the *morphisms from X to Y*. If $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, an alternative notation is $f : X \to Y$ or $X \xrightarrow{f} Y$.
- For every triple $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, a set map

$$7\,\mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{C}}(X, Z)$$

  called *composition*. (If $g : Y \to Z$ and $f : X \to Y$, write gf or $g \circ f : X \to Z$.)

satisfying the axioms:

1. $\mathrm{Hom}(X, Y) \cap \mathrm{Hom}(X', Y') = \emptyset$ unless $X = X'$ and $Y = Y'$.
2. For each $X \in \mathrm{Ob}\,\mathscr{C}$, there is an *identity morphism* $1_X \in \mathrm{Hom}(X, X)$, such that for any $Y, Z \in \mathrm{Ob}\,\mathscr{C}$, $f \in \mathrm{Hom}(Y, X)$, and $g \in \mathrm{Hom}(X, Z)$, we have $1_X \circ f = f$ and $g \circ 1_X = g$.
3. Associativity of composition: $(h \circ g) \circ f = h \circ (g \circ f)$ whenever the compositions make sense.

Moreover, $f : X \to Y$ is an *isomorphism* if there is some $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

Examples of categories

1. **Set**

   - Objects = all sets.
   - Morphisms: $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ = all set functions from X to Y.
   - Composition = ordinary composition of functions[1]

   Note that an *isomorphism* in the category of sets is just any bijection.

2. **CRng**

   - Objects = Commutatitve rings with unity (i.e. what we usually just call rings)
   - Morphisms = ring homomorphisms

3. For a fixed ring R, ${}_R\mathbf{Mod}$:

   - Objects = R-modules.
   - Morphisms = R-module homomorphisms (also called R-*linear maps*). We write $\mathrm{Hom}_R(M, N) := \mathrm{Hom}_{{}_R\mathbf{Mod}}(M, N)$.

4. For a fixed ring A, ${}_A\mathbf{Alg}$:

   - Objects = A-*algebras*. By definition, an A-algebra is a ring R together with a fixed ring homomorphism $\alpha_R : A \to R$, which in practice is usually omitted from the notation.
   - $\mathrm{Hom}_{{}_A\mathbf{Alg}}(R, S) :=$ A-*algebra homomorphisms* from R to S. By definition, an A-algebra homomorphism from R to S is a map $f : R \to S$ such that $f \circ \alpha_R = \alpha_S$.

5. **Group**

   - Objects = groups
   - Morphisms = group homomorphisms

6. **Ab**: abelian groups.

[1] In categories which have a forgetful functor to the category of sets (see below), composition will be ordinary composition of set functions unless otherwise noted.

7. Any poset $(P, \leq)$ can be considered a category $\mathscr{C}$.

   - $\text{Ob}\,\mathscr{C} = P$. That is, the *elements* of P are the objects.
   - $\text{Hom}_{\mathscr{C}}(s,t)$ is empty unless $s \leq t$, in which case we say $\text{Hom}_{\mathscr{C}}(s,t)$ has one element (which you can call st if you want).
   - Composition in $\mathscr{C}$ is forced.

   In $\mathscr{C}$, the only isomorphisms are the identity maps.

   Conversely, if $\mathscr{D}$ is a category where the objects form a set, the sets $\text{Hom}_{\mathscr{D}}(X,Y)$ have cardinality at most 1, and the only isomorphisms are the identity maps, then $\mathscr{D}$ has a poset structure (which really describes its whole structure): Simply declare that $X \leq Y$ iff $\text{Hom}_{\mathscr{D}}(X,Y) \neq \emptyset$.

8. **Top**

   - Objects = Topological spaces[2]
   - Morphisms = continuous maps

9. Let $(S, \cdot)$ be a semigroup with identity 1_S. The structure of S can be considered a category $\mathscr{C}$:

   - Objects: $\mathscr{C}$ has just one object. (Say $\text{Ob}(\mathscr{C}) = \{A\}$.)
   - $\text{Hom}_{\mathscr{C}}(A,A) = S$. That is, the *elements* of S are the *morphisms*.
   - Composition is the multiplication in S.

   Note that the condition for S to be a group can be stated in terms of the category $\mathscr{C}$: S is a group iff every morphism in $\mathscr{C}$ is an isomorphism.

10. Subcategories:

    Let $\mathscr{C}, \mathscr{D}$ be categories. One says that $\mathscr{C}$ is a *subcategory* of $\mathscr{D}$ if $\text{Ob}(\mathscr{C}) \subseteq \text{Ob}(\mathscr{D})$ and for every pair $X, Y \in \text{Ob}(\mathscr{C})$, $\text{Hom}_{\mathscr{C}}(X,Y) \subseteq \text{Hom}_{\mathscr{D}}(X,Y)$. If this second containment is an equality for every pair of $\mathscr{C}$-objects, we say that $\mathscr{C}$ is a *full* subcategory of $\mathscr{D}$.

    For example, **Ab** is a full subcategory of **Group**. The category of sets and injective set maps is a subcategory of **Set** which is not full.

[2]Recall that a topological space is a pair (T, σ), where T is a set and $\sigma \subseteq 2^T$, the so-called *closed sets* of T, in such a way that $\emptyset \in \sigma$, $T \in \sigma$, and σ is closed under arbitrary intersections and finite unions. A *continuous map* $f : (S, \sigma_S) \to (T, \sigma_T)$ is a set map $S \to T$ in such a way that for any $C \in \sigma_T$, $f^{-1}(C) \in \sigma_S$.

11. For any category $\mathscr{C}$, the *opposite category* $\mathscr{C}^{\text{op}}$ is defined as follows:
    - $\text{Ob}(\mathscr{C}^{\text{op}}) := \text{Ob}\,\mathscr{C}$.
    - For any pair $X, Y \in \text{Ob}(\mathscr{C})$, $\text{Hom}_{\mathscr{C}^{\text{op}}}(X, Y) := \text{Hom}_{\mathscr{C}}(Y, X)$. That is, *all arrows are reversed.*
    - Composition in $\mathscr{C}^{\text{op}}$ is the reverse of composition in $\mathscr{C}$. That is, $g \circ_{\mathscr{C}^{\text{op}}} f := f \circ_{\mathscr{C}} g$.

12. For any pair $\mathscr{C}, \mathscr{D}$ of categories, the *product category* $\mathscr{C} \times \mathscr{D}$ is defined as follows:
    - $\text{Ob}(\mathscr{C} \times \mathscr{D}) := \text{Ob}(\mathscr{C}) \times \text{Ob}(\mathscr{D})$. That is, an object is a pair (C, D), where $C \in \text{Ob}(\mathscr{C})$ and $D \in \text{Ob}(\mathscr{D})$.
    - $\text{Hom}_{\mathscr{C} \times \mathscr{D}}((C, D), (C', D')) := \text{Hom}_{\mathscr{C}}(C, C') \times \text{Hom}_{\mathscr{D}}(D, D')$.
    - Composition is defined component-wise: $(g, g') \circ (f, f') := (g \circ f, g' \circ f')$.

    For any set S and any S-indexed set of categories $\{\mathscr{C}_s \mid s \in S\}$, one can similarly define the product category $\prod_{s \in S} \mathscr{C}_s$. If $\mathscr{C} = \mathscr{C}_s$ for all $s \in S$, we also write $\mathscr{C}^S$.

Definition 2. For categories $\mathscr{C}, \mathscr{D}$, a *(covariant) functor*[3] $F : \mathscr{C} \to \mathscr{D}$ consists of a function $F : \text{Ob}\,\mathscr{C} \to \text{Ob}\,\mathscr{D}$, and for every pair $X, Y \in \text{Ob}\,\mathscr{C}$ a function $F : \text{Hom}_{\mathscr{C}}(X, Y) \to \text{Hom}_{\mathscr{D}}(F(X), F(Y))$, such that $F(1_X) = 1_{F(X)}$ for all $X \in \text{Ob}\,\mathscr{C}$, and that whenever the composition $g \circ f$ in $\mathscr{C}$ makes sense, we have $F(g \circ f) = F(g) \circ F(f)$.

A *contravariant functor* $F : \mathscr{C} \to \mathscr{D}$ is a covariant functor $F : \mathscr{C} \to \mathscr{D}^{\text{op}}$.

Examples of functors

1. For any category $\mathscr{C}$, the *identity functor* $1_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is defined to be the identity map on objects and morphism sets of $\mathscr{C}$.

2. Abelianization $F : \textbf{Group} \to \textbf{Ab}$. For a group G, $F(G) := G/[G, G]$. For a group homomorphism $f : G \to H$, from group theory we know that there is an induced homomorphism $\tilde{f} : G/[G, G] \to H/[H, H]$ of abelian groups. Set $F(f) := \tilde{f}$. Then F is a functor.

[3]If it is unspecified whether a functor is covariant or contravariant, typically assume it is covariant.

3. For *any* two functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ (covariant or contravariant), the *composition* $G \circ F : \mathscr{C} \to \mathscr{E}$ is a functor if we set $(G \circ F)(C) := G(F(C))$ on objects and $(G \circ F)(f) := G(F(f))$ on morphisms. If F and G are both covariant *or* both contravariant, then $G \circ F$ is covariant. If one is covariant and one contravariant, then the composition is contravariant.

4. Forgetful funtors:

   If $\mathscr{C}$ is a category whose objects have underlying sets and whose morphisms are underlying set maps (e.g. **Group**, **Ab**, **CRng**, **Top**, ${}_A\mathbf{Alg}$, or ${}_R\mathbf{Mod}$), then there is a *forgetful functor* $U : \mathscr{C} \to \mathbf{Set}$, which sends each object to its underlying set and each morphism to its underlying set map. Note that in the same way, there are also sequences of "forgetful functors" ${}_R\mathbf{Alg} \to \mathbf{CRng} \to \mathbf{Ab}$ and ${}_R\mathbf{Alg} \to {}_R\mathbf{Mod} \to \mathbf{Ab} \to \mathbf{Group}$.

   In general, a forgetful functor is a functor $U : \mathscr{D} \to \mathscr{C}$ such that the objects and/or morphisms of $\mathscr{D}$ have strictly more algebraic structure than those of $\mathscr{C}$, and U is the map which corresponds to removing the "extra" structure from the objects and morphisms.

5. Diagonal functors:

   If $\mathscr{C}$ is a category, the *diagonal functor* $\Delta : \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ is defined on objects by $\Delta(C) := (C, C)$ and on morphisms by $\Delta(f) = (f, f)$. If S is a set, one can similarly define the diagonal functor $\mathscr{C} \to \mathscr{C}^S$.

6. The prime spectrum functor – Spec : $\mathbf{CRng} \to \mathbf{Top}^{\text{op}}$.

   We will next define a functor which is of particular importance to commutative ring theory. It will take some development (see below):

Spec as a contravariant functor

Thus far, we have defined $\operatorname{Spec} R$ as the *set* of prime ideals of R. There is, however, a useful way to define a topology on it.

Topology of Spec

Fix the ring R for the moment, and set $X := \operatorname{Spec}(R)$. For any ideal $I \subseteq R$, define $V(I) := \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq I\}$. Then set $\sigma_X := \{V(I) \mid I \subseteq$

R an ideal$\}$. These will be the closed sets for our topology on X.

To see that σ_X is allowable as the collection of closed sets for a topology, we have that

- $V(0) = X$,
- $V(R) = \emptyset$,
- For any two ideals I, J of R, define the *product* IJ to be the set of *finite sums* of elements of R of the form ij, where $i \in I$ and $j \in J$. (Prove that IJ is an ideal!) Then we have $V(I) \cup V(J) = V(IJ)$. (prove this!)
- For any set $\{I_\alpha \mid \alpha \in S\}$ of ideals indexed by a set S, define the *sum* $J = \sum_{\alpha \in S} I_\alpha$ to be the set:

  $$J := \{\sum_{i=1}^{n} a_i \mid n \in \mathbb{N}, a_i \in I_{\alpha_i}, \alpha_i \in S\}$$

  That is, J is the set of all finite sums of elements taken from the ideals $\{I_\alpha\}$ (prove that J is an ideal!). Then $\bigcap_{\alpha \in S} V(I_\alpha) = V(\sum_{\alpha \in S} I_\alpha)$. (prove this!)

Hence, $X = (X, \sigma_X)$ is a topological space. From now on when we refer to $\operatorname{Spec} R$, we refer to this topology.

The functor Spec

We have now defined Spec as a map between *objects* of **CRng** and $\mathbf{Top}^{\mathrm{op}}$. Now we need to define it on *morphisms*.

Let $\phi : R \to S$ be a ring map. Set $(\operatorname{Spec} \phi)(P) := \phi^{-1}(P)$. This is a set map because if P is prime, so is $\phi^{-1}(P)$, since $R/\phi^{-1}(P) \hookrightarrow S/P$ is an injective map of rings and S/P is an integral domain, hence $R/\phi^{-1}(P)$ is also a domain. (Prove that $\operatorname{Spec} \phi$ is not just a set map, but a continuous map!) Hence, Spec is a contravariant functor.

What about maximal ideals?

Recall that $\Omega(R)$ was defined to be the set of maximal ideals of R. We can topologize $\Omega(R)$ by giving it the subspace topology from the set inclusion $\Omega(R) \subseteq \operatorname{Spec}(R)$. However, in general defining Ω in this way does *not* give a functor, because the inverse image of a maximal ideal is often not maximal. (e.g. for the injective ring map $\mathbb{Z} \to \mathbb{Q}$, 0 is maximal in $\mathbb{Q}$ but not in $\mathbb{Z}$.)

However, as we shall see later, if k is a field and R, S are both finitely generated k-algebras, the inverse image of a maximal ideal of S is always a maximal ideal of R, and indeed Ω does give a contravariant functor from the category of finitely generated k-algebras to the category of topological spaces. This fact is basic to the interaction between commutative algebra and classical algebraic geometry.