

Errata for “Regular Neighbourhoods and Canonical
Decompositions for Groups”, Asterisque 289 (2003), by Peter
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There are two errors in the discussion of our construction of algebraic regular neighbourhoods. While they are easy to fix, this requires changes to the paper in several places. We discuss these errors first. Then we list the remaining errors in order of their occurrence.

Invertible almost invariant sets

The first problem is in our treatment of almost invariant sets which are invertible (see Definition 2.12). Before discussing the details, we need to briefly recall the construction in chapter 3. We have a finitely generated group G with finitely generated subgroups H_1, \dots, H_n and, for $1 \leq i \leq n$, we have a nontrivial H_i -almost invariant subset X_i of G . Recall that E denotes the set of all translates of all the X_i 's and X_i^* 's, and that an element U of E is isolated if it crosses no element of E . We construct the algebraic regular neighbourhood $\Gamma(X_1, \dots, X_n)$ in stages. First we consider the case when the X_i 's are in good position. This means that if U and V are any elements of E and two of the four sets $U^{(*)} \cap V^{(*)}$ are small, then one must be empty. In Theorem 3.8, we describe a graph of groups structure $\Theta(X_1, \dots, X_n)$ for G . If no X_i is simultaneously isolated and equivalent to an invertible almost invariant set, then $\Gamma(X_1, \dots, X_n)$ is defined to be $\Theta(X_1, \dots, X_n)$. If some X_i is isolated and is equivalent to an invertible almost invariant set Y_i , we claimed (see the last five lines of page 48) that we could replace each such X_i by an equivalent almost invariant set Z_i such that Z_i is not invertible and the new family is in good enough position. (Good enough position means that if U and V are any elements of E and two of the four sets $U^{(*)} \cap V^{(*)}$ are small, then either one must be empty, or some element of E crosses both U and V .) This allowed us to define $\Gamma(X_1, \dots, X_n)$ in general. Unfortunately this claim is incorrect. The error is in the existence part of Lemma 3.14 and is already clear in the case when $n = k = 1$.

Here is the statement of the existence part of Lemma 3.14 in this case.

Let G denote a finitely generated group, and let H be a finitely generated subgroup of G . Let X be a nontrivial H -almost invariant subset of G , such that X is isolated. Then there is an almost invariant set Z equivalent to X , such that Z is not invertible, and is in good enough position.

The following example shows that this statement is false.

Example Let G denote the group $\mathbb{Z}_2 * \mathbb{Z}_2$, let H denote the trivial subgroup and let σ denote the given splitting of G over H . Let X be a H -almost invariant subset of G associated to σ . As X is associated to a splitting, it crosses none of its translates and so is isolated in the set $E(X)$ which consists of all translates of X and X^* . Let Z be an almost invariant set which is equivalent to X . If Z is in good enough position, we claim that Z must be invertible. Thus we have the required counterexample to the above statement.

Here is the proof of the claim that Z must be invertible. The key point is that every translate of Z is equivalent to Z or Z^* , so that two translates of Z never cross. Thus the fact that Z is in good enough position implies that Z is in good position. It follows that $E(Z)$, the set of all translates of Z and Z^* , is nested. Now we can use Dunwoody's construction to produce a G -tree T whose oriented edges correspond to the elements of $E(Z)$. As Z is equivalent to X which is associated to the given splitting σ , the splitting of G determined by T must be conjugate to σ . In particular, the quotient graph of groups $G \backslash T$ has one edge and two vertices, with the edge group being trivial and the vertex groups having order 2. Thus T is a copy of the real line and the elements in G of order 2 act on T by reflections. Further each vertex of T is the fixed point of one of these reflections. Let s denote the oriented edge of T which corresponds to Z , choose the terminal vertex w of s as the basepoint, and let $\varphi : G \rightarrow V(T)$ denote the G -equivariant map such that $\varphi(e) = w$. Then Z equals Z_s . Let v denote the second vertex of s . As v does not lie in the G -orbit of w , it follows that $\varphi^{-1}(v)$ is empty. If k denotes the element of G which acts on T by a reflection fixing v , then clearly $kZ_s = Z_s^*$ so that $kZ = Z^*$. Thus Z is invertible as claimed.

Here is the way to resolve this problem. If some isolated X_i is equivalent to an invertible almost invariant set Y_i , we need to make slight adjustments at each stage of our construction. The above example shows that we may not be able to find almost invariant sets Z_i equivalent to X_i , such that the Z_i 's are in good enough position and not invertible. Thus we will not be able to construct our algebraic regular neighbourhood of the X_i 's as described on pages 46–50. Instead of replacing each X_i by such a non-invertible set Z_i , we replace each X_i by an equivalent almost invariant set Y_i so that the Y_i 's are in good enough position. This can be done by Lemma 3.13. Then we simply construct the graph of groups $\Theta(Y_1, \dots, Y_n)$. If no Y_i is isolated and

invertible, then this is $\Gamma(X_1, \dots, X_n)$. If some Y_i is isolated and invertible, we subdivide the corresponding edge of $\Theta(Y_1, \dots, Y_n)$, as discussed in Examples 3.10 and 3.11 on pages 44 and 45, and take $\Gamma(X_1, \dots, X_n)$ to be the resulting graph of groups.

This involves corresponding changes in the statements and proofs at several points in chapters 3, 4, 5 and 6:

Pages 50-51: the statement of Summary 3.16.

Pages 51-52: the proof of Lemma 3.17.

Pages 53-54: the discussion of the construction of an algebraic regular neighbourhood of an infinite family.

Page 62: the proof of part 2) of Lemma 4.10.

Page 67: Lemma 5.1 and its proof are correct, but the comment just before Lemma 5.1 that "it suffices to prove that the V_0 -vertices of $\Gamma(\{X_\lambda\}_{\lambda \in \Lambda} : G)$ enclose the given X_λ 's in the case when the X_λ 's are in good position and isolated X_λ 's are not invertible" is not correct. Thus Lemma 5.1 should be stated and proved without this last assumption.

Page 75: the proof of Lemma 5.10.

Page 78: the proof of Theorem 5.16.

Pages 80-83: the discussion of the construction of an algebraic regular neighbourhood of a family of almost invariant sets over groups which need not be finitely generated.

Page 89: the proof of Theorem 6.6.

Splittings over non-finitely generated groups

The second problem is in our treatment of splittings over non-finitely generated groups. Before discussing the details, we need some background discussion.

Let G be a finitely generated group. A HNN extension σ of G is said to be *ascending* if $G = A*_C$ where at least one of the two injections of C into A is an isomorphism. Note that if both injections are isomorphisms, then A is normal in G with infinite cyclic quotient. The difficulty arises when one considers ascending HNN extensions $G = A*_C$ for which A is not finitely generated. Such extensions are not at all unusual, as all that is needed is a surjection from G to \mathbb{Z} whose kernel A is not finitely generated. Whether or not A is finitely generated, ascending HNN extensions have the property that they can only be compatible with ascending HNN extensions. (Recall that two splittings σ and τ of a group G are compatible if G is the fundamental group of a graph of groups with two edges such that the associated edge splittings are σ and τ .) The precise result is the following.

Lemma 1 *Let σ be an ascending HNN extension of a group G . If σ is compatible with a splitting τ of G , then τ is also an ascending HNN extension.*

Proof. If σ is compatible with τ , then G is the fundamental group of a graph Γ of groups such that Γ has two edges s and t and the associated edge splittings are σ and τ respectively. As σ is HNN, it follows that Γ has at most two vertices. We claim that Γ has two vertices each of valence 2, so that Γ is a circle. For otherwise, the edge s would be a loop, and the subgraph given by t would carry the vertex group A of σ . But this is impossible as one of the inclusions of the edge group of s into A is an isomorphism. Now we know that Γ is a circle, it is easy to see that τ is also an ascending HNN extension. ■

Note that compatible ascending HNN extensions need not be conjugate. To construct examples, let K be a subgroup of a group H such that K itself has a subgroup L isomorphic to H . Let G be the fundamental group of a graph of groups with underlying graph a circle, with two vertices labeled by H and K and two edges labeled by K and L , so that the inclusion of the edge group L into the vertex group H is an isomorphism and the other inclusions are clear. If K and L are not isomorphic, the two edge splittings cannot be conjugate. A simple such example can be found with H (and hence L) free of rank 2, and K free of rank 3.

The problem in our arguments is a technical one which relates to the question of when two splittings are compatible. In particular, we are grateful to Vincent Guirardel for showing us an example which demonstrates that Theorem 5.16 is incorrect. His example consists of two splittings which have intersection number zero but are not compatible. Both splittings are ascending HNN extensions over non-finitely generated groups. However such splittings are the only source of problems, and we can correct our development of the theory of algebraic regular neighbourhoods fairly easily. This requires many changes to the paper including a modification of the definition of an algebraic regular neighbourhood (Definition 6.1).

Here is Guirardel's example. Let G denote the free group on two generators a and b , and let $f : G \rightarrow \mathbb{Z}$ be given by $f(a) = 0$, and $f(b) = 1$. Let K denote the kernel of f , so that K is freely generated by the elements $b^k ab^{-k}$, $k \in \mathbb{Z}$. Let τ denote the splitting of G as $K *_K$, where both injections of the edge group into the vertex group are isomorphisms. Let H denote the subgroup of G generated by the elements $b^k ab^{-k}$, $k \geq 0$, and let σ denote the splitting of G as $H *_H$, in which one inclusion is the identity and the other is conjugation by b . We will show that the splittings σ and τ have intersection number zero and cannot be compatible.

Let S and T denote the G -trees corresponding to σ and τ respectively. Thus T is a copy of the real line, with a vertex at each integer point. Think of T as the x -axis in the plane with S above it. We can describe S pictorially by saying that each vertex of S has integer x -coordinate, no edge of S is vertical, and at each vertex v of S , there is exactly one edge incident to v from the right. Now the projection of the plane onto the x -axis induces a G -equivariant map $p : S \rightarrow T$ which induces the identity on G . Let s be an edge of S and let t denote the edge $p(s)$ of T , and orient s and t to point to the left. Choose a base vertex $*$ for S and let $p(*)$ be the base vertex of T . Thus we have $\varphi : G \rightarrow V(S)$ given by $\varphi(g) = g(*)$, for all $g \in G$. If we remove the interior of s from S , we are left with two subtrees of S . The one which contains the terminal vertex of s is denoted by Y_s , and we let Z_s denote $\varphi^{-1}(Y_s)$. Similarly removing the interior of t from T yields two half lines in T , and the one which contains the terminal vertex of t is denoted by Y_t , and we let Z_t denote $\varphi^{-1}(p^{-1}(Y_t))$. The sets Z_s and Z_t are almost invariant subsets of G over H and K respectively which are associated to the splittings σ and τ of G . Clearly $Z_s \subset Z_t$. As gZ_t is equivalent to Z_t for every g in G , we have $Z_s < gZ_t$, for every g in G . Thus σ and τ have intersection number zero. Now consider the set E of all translates of Z_s and Z_t and their

complements in G . If σ and τ were compatible, E would correspond to the edges of a G -tree. In turn this would imply that there must some translate of Z_t (or of Z_t^*) which is $< Z_s$. As $Z_s < gZ_t$, for every g in G , this is clearly impossible.

It is interesting to see how our construction of an algebraic regular neighbourhood fails in this case. As no two elements of E cross, each element of E is isolated in E and so forms a CCC by itself. It follows immediately that the CCC's of E form a pretree. The problem is that this pretree is not discrete. For if b acts on T by translating one unit to the left, then we have the inclusions $\dots \subset b^2Z_t \subset bZ_t \subset Z_t$ and $Z_s < b^kZ_t$ for each $k \geq 0$. Thus there are infinitely many CCC's of E between Z_s and Z_t .

The error which causes all the problems occurs in the proof of Proposition 5.7. On page 72, lines -13 to -12, we assert that “there must be an element g of G such that $gX \subset U$.” This is not correct in general. In order to appreciate the problem, we need to discuss the proof of Proposition 5.7. Recall that we have a family $\{X_\lambda\}$ of almost invariant subsets of G in good position such that the regular neighbourhoods of this family can be constructed as in chapter 3. Recall also that E denotes the collection of all translates of the X_λ 's and X_λ^* 's, and that X is a nontrivial H -almost invariant subset of G which crosses no element of E . Proposition 5.7 asserts that X is enclosed by a V_1 -vertex of Γ so long as either H is finitely generated or X is associated to a splitting of G . Our proof starts by showing that X is sandwiched between two elements of E , i.e. there are elements U and V of E such that $U < X < V$. If this condition holds then the remainder of our proof is correct. Further our proof that X must be sandwiched between two elements of E is correct, so long as H is finitely generated. In the case when H is not finitely generated, so that X is associated to a splitting σ of G , our proof is also correct so long as σ is not an ascending HNN extension. This was the case which we discussed incorrectly on page 72, line -12. For convenience in what follows we will say that a splitting of a group which is an ascending HNN extension over a non-finitely generated group is *special*.

Given a collection E of almost invariant subsets of G , we will say that an almost invariant subset X of G is *sandwiched by E* if there are elements U and V of E such that $U \leq X \leq V$. We can now correct the statement of part 2) of Proposition 5.7 by simply adding the hypothesis that X be sandwiched by E . Our published proof of Proposition 5.7 shows that this is automatic except possibly when X is associated to a special splitting of

G. This weakens Proposition 5.7, but almost all of our applications will still follow from this weakened version. Note that the proof of this weakened form of Proposition 5.7 is substantially shorter.

Here is a list of consequential changes:

Lemmas 5.10 and 5.15 need an extra sandwiching assumption. In the last paragraph of the proof of Lemma 5.10, on page 75, our arguments use the assumptions that A is sandwiched between two elements of E and that any element of E is sandwiched between two translates of A and A^* . The first assumption follows from the hypotheses of the lemma, but the second does not and needs to be added to the list of hypotheses. Of course, the second assumption is correct unless some edge splitting of Γ is special, and in this case, we know that Γ must be a circle. The same point arises on page 77 in the proof of Lemma 5.15.

The existence part of Theorem 5.16 is correct so long as we exclude special splittings. Otherwise the two splittings σ and τ described above form a counterexample. But the uniqueness part of Theorem 5.16 is correct, so that Theorem 5.17 is still correct.

Lemma 5.19 is correct.

Lemma 5.21 is also correct. We note that the assumption in this lemma and several later results that the regular neighbourhood has been constructed as in chapter 3 avoids many of the above worries about sandwiching. In particular, it implies that if P denotes the pretree of CCC's of E , the set of all translates of all the X_λ 's and X_λ^* 's, then P is discrete.

On pages 80-83, in the discussion of algebraic regular neighbourhoods of almost invariant sets over possibly non-finitely generated groups, we need to add a sandwiching assumption in order to prove existence for finite families. For example, we could assume that each X_i is sandwiched by $E(X_j)$, for each i and j .

Proposition 5.23 also needs an additional sandwiching assumption.

In part 2) of the definition of an algebraic regular neighbourhood (Definition 6.1), we need to add the assumption that the almost invariant set associated to the splitting σ is sandwiched by E .

Our main existence result (Theorem 6.6) is fine if the H_i 's are all finitely generated. In general, we need to add a sandwiching assumption. For example, we could assume that each X_i is sandwiched by $E(X_j)$, for each i and j .

Our main uniqueness result (Theorem 6.7) is correct as stated. The proof

uses Lemma 5.10 crucially, but the sandwiching assumption which needs to be added to the statement of Lemma 5.10 holds automatically in the present context.

The change in the definition of an algebraic regular neighbourhood means that we need to change the statements of many later theorems in which we describe the properties of the regular neighbourhoods we construct. For example, part (5) of Theorem 9.4 needs the addition of a sandwiching assumption. Note that we also used Proposition 5.7 to prove part (9) of Theorem 9.4, but this needs no change as the splittings considered are over finitely generated subgroups of G . The same comments apply to all the similar later results.

Having discussed the two errors in the construction of an algebraic regular neighbourhood, we now continue this list of errata in order of occurrence.

Page 28: There is an error in the second sentence of Example 2.34. In this example, G denotes the free group of rank 2. The sentence in question asserts that if C is a subgroup of G which is not finitely generated, then $e(G, C) = \infty$. This is incorrect. For example, if C is the kernel of the abelianisation map $G \rightarrow \mathbb{Z}^2$, then $e(G, C) = e(\mathbb{Z}^2) = 1$. However this does not invalidate Example 2.34. It is easy to see that there are many subgroups C of G which are not finitely generated for which $e(G, C) = \infty$. For a specific example, let H be a subgroup of G of index 2, so that H is free of rank 3, and let C denote the kernel of some surjection from H to the free group F_2 of rank 2. Then $e(G, C) = e(H, C) = e(F_2) = \infty$.

Note that the argument in Example 2.34 does not need $e(G, C) = \infty$. All that is needed is that $e(G, C) > 1$ in order that there be a nontrivial C -almost invariant subset of G .

Page 103: Theorem 7.11 is incorrect as stated, and this requires some minor modifications in the proofs of the applications of this result.

The wording of the statement needs to be modified. Let Γ_1 and Γ_2 be graphs of groups decompositions of a group G . We will say that Γ_2 is a *proper* refinement of Γ_1 if it is obtained from Γ_1 by splitting at a vertex so that the induced splitting of the vertex group is nontrivial. A sequence of proper refinements will also be called a proper refinement. Our proof of Theorem 7.11 assumes that Γ_{k+1} is a proper refinement of Γ_k . This assumption is implicit in the second sentence of the proof, where we assert that we have only to bound the length of chains of splittings of G over descending subgroups.

All of the above comments apply also to Theorem 7.13.

The first paragraph of the proof of Theorem 8.2 uses Theorem 7.11, and this paragraph needs changing as follows. Lines 1-6 of this paragraph are fine. But the next sentence is incorrect. It should assert that if G possesses a splitting σ' over a two-ended subgroup C' commensurable with H which has intersection number zero with the edge splittings of \mathcal{G} , then this splitting is enclosed by some V_0 -vertex v of \mathcal{G} , and determines a trivial splitting of the vertex group $G(v)$. This does not imply that σ' is conjugate to one of the edge splittings of \mathcal{G} . This requires changing the proof of Theorem 8.2 on lines 17-18 of page 111. Our choice of \mathcal{G} does not imply that σ must be conjugate to one of the edge splittings of \mathcal{G} , as claimed, but it does imply

the result of the next sentence which is all we need.

At the end of the proof of Proposition 8.4, we quote Theorem 7.11 to show that one cannot have an unbounded chain σ_i of compatible splittings of G over strictly descending subgroups, and this does follow from the corrected version of Theorem 7.11. The use of Theorem 7.11 in the proof of Theorem 9.2 is correct as Γ_{i+1} is a proper refinement of Γ_i for each i .

Theorems 7.11 and 7.13 are used in several other places. The key point to note is that the only problem which might occur is the existence of an unbounded chain σ_i of compatible splittings of G over strictly ascending subgroups H_i . If this occurs and all the H_i 's are *VPC* of the same length, then they are all commensurable, and hence each H_i has large commensuriser. In particular, when constructing an algebraic regular neighbourhood, the splittings σ_i must all be enclosed by a single V_0 -vertex of large commensuriser type.

Page 132: Theorem 10.8 is false. This clarifies why it is so important for our theory to consider *all* almost invariant subsets not just those which correspond to splittings. The generalisations of Theorem 10.8 in Theorems 12.6, 13.15 and 14.11 are also false. We will describe a counterexample.

Let K denote the Baumslag-Solitar group $BS(1, 2) = \langle a, t : tat^{-1} = a^2 \rangle$. Thus K has a natural expression as a HNN extension with infinite cyclic vertex group generated by a . It follows that K is finitely presented and torsion free. The map from K to the integers \mathbb{Z} given by killing a has kernel C which is isomorphic to the additive group of the dyadic rationals $\mathbb{Z}[\frac{1}{2}]$. If we choose the isomorphism so that a corresponds to 1, then $t^{-1}at$ corresponds to $\frac{1}{2}$ and in general $t^{-n}at^n$ corresponds to $\frac{1}{2^n}$. We let A denote the cyclic subgroup of C generated by a and let A_n denote the cyclic subgroup of C generated by $t^{-n}at^n$. Thus $A = A_0$ and the union of all the A_n 's equals C . It is easy to see that K admits no free product splitting. As K is torsion free, it follows that K admits no splitting over any finite subgroup and so is one-ended. Next we let G denote the group $K \times \mathbb{Z}$. Then G admits no splitting over any *VPCk* subgroup, for $k \leq 1$. Let H be an isomorphic copy of G , let D , B and B_n denote the subgroups of H which correspond to C , A and A_n respectively, and define \overline{G} to be $G *_A H$, the double of G along A . Note that \overline{G} is also finitely presented, and it is easy to see that \overline{G} is also one-ended. Thus the regular neighbourhood $\Gamma_1(\overline{G})$ exists. Note that \overline{G} commensurises A so that A has large commensuriser in \overline{G} .

Now \overline{G} admits splittings σ_i over A_i and τ_i over B_i which can be described

in the following simple way. We define σ_i by writing

$$\overline{G} = G *_A H = (G *_A A_i) *_A H = G *_A (A_i *_A H),$$

and define τ_i by writing

$$\overline{G} = G *_B H = G *_B (B_i *_B H) = (G *_B B_i) *_B H.$$

Note that $\sigma_0 = \tau_0$. It is easy to see that all these splittings are compatible. In fact the definitions already show that σ_i and σ_0 are compatible and that τ_i and τ_0 are compatible. The lemma below states that these are the only splittings of \overline{G} over a two-ended subgroup, up to conjugacy. It follows that the family \mathcal{S}_1 of all splittings of \overline{G} over two-ended subgroups contains an infinite collection of isolated splittings of \overline{G} , which shows that \mathcal{S}_1 cannot have a regular neighbourhood. Thus Theorems 10.8, 12.6, 13.15 and 14.11 are all false.

Lemma 2 *Any splitting of \overline{G} over a two-ended subgroup is conjugate to some σ_i or τ_j .*

Proof. Suppose that \overline{G} has a splitting σ over a two-ended subgroup. Let T be the corresponding \overline{G} -tree, so that $\overline{G} \backslash T$ has a single edge. Recall that G admits no splitting over any $VPCk$ subgroup, for $k \leq 1$. It follows that the subgroups G and H of \overline{G} must each fix a vertex of T . These vertices must be distinct as otherwise \overline{G} itself would fix a vertex of T . Also note that G and H can each fix only a single vertex of T , as otherwise G or H would fix some edge of T and hence itself be $VPCk$, for $k \leq 1$, which is not the case. Let v and w denote the vertices fixed by G and H respectively, and let λ denote the edge path in T which joins them. As G and H together generate \overline{G} , it follows that σ is an amalgamated free product. Further, by considering canonical forms of elements, it is easy to see that λ must consist of a single edge. Thus $\overline{G} = P *_R Q$, where P , Q and R denote the stabilisers of v , w and λ respectively. Note that this splitting of \overline{G} is conjugate to σ . As $G \subset P$ and $H \subset Q$, it follows that $G \cap H = A \subset R$. As R and A are each two-ended, it follows that R contains A with finite index. In particular we have the well known fact that R must be conjugate into G or into H . Now consider the subgroup GR of P generated by G and R . If R is contained in G , this subgroup equals G . Otherwise the fact that R lies in some conjugate of G or H implies that $GR = G *_A R$. Similarly the subgroup HR of Q generated by

H and R is equal either to H or to $R *_A H$. Now the inclusions $GR \subset P$ and $HR \subset Q$ induce a natural injection of $GR *_R HR$ into $\overline{G} = P *_R Q$. As G and H together generate \overline{G} , this injection must also be surjective and hence an isomorphism. It follows that $GR = P$ and $HR = Q$. If R lies in G and in H , the fact that R contains A implies that $R = A$. In this case $GR = G$ and $HR = H$, and the splitting σ is conjugate to σ_0 . If R lies in G , the facts that G is isomorphic to $K \times \mathbb{Z}$ and R contains A with finite index imply that R must equal A_n , for some $n \geq 0$. In this case $GR = G$ and $HR = A_n *_A H$, and the splitting σ is conjugate to σ_n . Similarly if R lies in H , then R must equal B_n , for some $n \geq 0$, and in this case the splitting σ is conjugate to τ_n . This leaves the case when R is not contained in G or in H . In this case we have an isomorphism between $G *_A H$ and $(G *_A R) *_R (R *_A H) = G *_A R *_A H$. But this is impossible as we know that R is conjugate into G or H . The result follows. ■

Page 144: The statement of Theorem 12.3 is not quite correct. There is a special case when Γ_n consists of a single V_0 -vertex. In this case there is a possibility which is not mentioned in the statement. Namely G may be $VPC(n+1)$. This possibility is contained in the paper by Dunwoody and Swenson but we omitted it in error when we applied their results.

The same omission occurs in the statements of Theorems 12.5, 12.6, 13.12, 13.13, 14.5, and 14.6.

Page 161: Example 14.1 is wrong. Recall that in chapter 14, we gave a construction of the regular neighbourhood $\Gamma_{1,2,\dots,n}$ in which we restrict attention to almost invariant sets over virtually abelian subgroups. The point of Example 14.1 was to show that this construction does not work if one considers almost invariant sets over virtually polycyclic subgroups. After discussing the error in Example 14.1, we will give a new example which demonstrates the phenomenon which Example 14.1 was supposed to demonstrate. Thus it is still correct to say that the construction of $\Gamma_{1,2,\dots,n}$ does not work if one considers almost invariant sets over virtually polycyclic subgroups.

Example 14.1 is supposed to be an example of a one-ended group G with incommensurable polycyclic subgroups H and K of length 3, and 2-canonical almost invariant sets X and Y over H and K respectively which cross weakly. The sets X and Y described in the example do cross weakly but they are not 2-canonical, as we will now show.

Recall that G is constructed by amalgamating H , K and a third group L along a certain infinite cyclic subgroup C . Thus G has subgroups $H *_C K$, $H *_C L$, $K *_C L$. Further G can be expressed as the amalgamated free product of the first two groups over H , and X is an H -almost invariant subset of G associated to this splitting. Similarly, the first and third groups give an amalgamated free product decomposition of G over K , and Y is associated to this splitting. Recall that Γ denotes the graph of groups structure for G which is a tree with four vertices carrying the subgroups C , H , K and L such that each edge group is C . We will label the vertices c , h , k and l correspondingly. Note that all three edges are incident to c .

The error in our argument occurs where we claim that if W is a nontrivial almost invariant subset of G over a two-ended subgroup A , then W is enclosed by the vertex l of Γ which carries L . The two edge splittings given by the edges ch and ck of Γ are clearly not enclosed by l , and so the associated C -almost invariant subsets of G are also not enclosed by l .

Now we will show that X and Y are not 2-canonical. In fact, they are not 1-canonical. Recall that X is associated to the splitting σ of G as $(H *_C K) *_H (H *_C L)$. We claim that this splitting is not compatible with the splitting τ of G over C as $H *_C (K *_C L)$. This means that X crosses the C -almost invariant subset of G associated to τ , so that X is not 1-canonical. A similar argument shows that Y is also not 1-canonical. To prove our claim suppose that σ and τ are compatible. This implies that G is the fundamental group of a graph Γ' of groups which has two edges such that the associated edge splittings are σ and τ . As neither splitting is HNN, Γ' must be homeomorphic to an interval with vertices P and R at the endpoints and one interior vertex Q . Choose notation so that σ is the edge splitting associated to PQ , and that τ is the edge splitting associated to QR . Thus the edge group associated to QR is C . Also the group G_P must be $H *_C K$ or $H *_C L$, and the group G_R must be H or $K *_C L$. If G_R is H , the fact that in either case G_P also contains H implies that H is contained in each edge group of Γ' . But this is impossible as C cannot contain H , as C is cyclic and H is not. It follows that G_R must be $K *_C L$. But G_P contains one of K or L , which implies that K or L is contained in C which is again impossible. This completes the proof of the claim.

Here is an example of a one-ended, finitely presented group G which has 2-canonical splittings σ_1 and σ_2 over $VPC3$ subgroups C_1 and C_2 respectively, such that σ_1 and σ_2 cross weakly and C_1 and C_2 are not commensurable.

This replaces the incorrect Example 14.1.

Let B denote the free abelian group of rank 2, let θ be a hyperbolic automorphism of B (i.e. θ has two real eigenvalues with absolute value not equal to 1), and let C be the extension of B by \mathbb{Z} determined by θ . Thus B is normal in C with quotient isomorphic to \mathbb{Z} . Fix an element α of C such that conjugation of B by α induces the automorphism θ , and let A denote the infinite cyclic subgroup of C generated by α . Thus A projects onto the infinite cyclic quotient of C by B . A crucial property of C , which follows from the hyperbolicity of θ , is that any *VPC2* subgroup of C must be a subgroup of B . Let nA denote the subgroup of A of index n . As A is a maximal cyclic subgroup of C , it follows that $Z_C(nA) = A$, where $Z_C(X)$ denotes the centraliser in C of X . Note that C is the fundamental group of a closed 3-manifold M which is a torus bundle over the circle.

For any integer i , let B_i denote a copy of B , let θ_i denote the automorphism of B_i which corresponds to θ , let C_i denote the corresponding copy of C , and let α_i denote the corresponding element of C_i . For any pair i, j of distinct integers, let C_{ij} denote the extension of $B_i \oplus B_j$ by \mathbb{Z} , given by the automorphism $\theta_i \oplus \theta_j$. Fix an element α_{ij} of C_{ij} such that conjugation of $B_i \oplus B_j$ by α_{ij} induces the automorphism $\theta_i \oplus \theta_j$, and let A_{ij} denote the infinite cyclic subgroup of C_{ij} generated by α_{ij} . Thus there is a natural inclusion of C_i into C_{ij} which sends α_i to α_{ij} , and hence sends A_i to A_{ij} .

We note the following facts about the C_{ij} 's. Each C_{ij} is one-ended and torsion free, and any *VPC2* subgroup of C_{ij} must be a subgroup of $B_i \oplus B_j$. It follows that each C_{ij} has no nontrivial almost invariant subsets over any *VPC* subgroup of length ≤ 2 . It is easy to see that $Z_{C_{ij}}(nA_{ij}) = A_{ij}$.

We construct a group $H = C_{13} *_{C_1} C_{01} *_{C_0} C_{02} *_{C_2} C_{24}$, where the inclusions of the edge groups into the vertex groups are the natural ones. Thus we have in H the equations $\alpha_{13} = \alpha_1 = \alpha_{01} = \alpha_0 = \alpha_{02} = \alpha_2 = \alpha_{24}$, and we abuse notation and denote this element by α and the cyclic subgroup of H it generates by A . The natural homomorphisms $C_{ij} \rightarrow \mathbb{Z}$ all fit together to yield a homomorphism $H \rightarrow \mathbb{Z}$ which maps A onto \mathbb{Z} . We also choose a group D which is the fundamental group of a closed hyperbolic 3-manifold and has $H_1(D) \cong \mathbb{Z}$. Then D is one-ended and torsion free. It has no nontrivial almost invariant subsets over any *VPC1* subgroup and contains no *VPC2* subgroups. Thus D has no nontrivial almost invariant subsets over any *VPC* subgroup of length ≤ 2 . We define $G = H *_A D$, where A is identified with a cyclic subgroup of D which maps onto $H_1(D)$. Thus the homomorphism $H \rightarrow \mathbb{Z}$ extends to one from G to \mathbb{Z} which maps A onto \mathbb{Z} . As H and D are

both torsion free, so is G . Note that as A is a maximal cyclic subgroup of D , it follows that $Z_D(A) = A$.

The group H is the fundamental group of a graph Δ of groups with underlying graph an interval divided into three edges. The edge groups are C_1 , C_0 and C_2 , and we denote the corresponding edges by e_1 , e_0 and e_2 respectively. The vertex groups are C_{13} , C_{01} , C_{02} and C_{24} , and we denote the corresponding vertices by v_{13} , v_{01} , v_{02} and v_{24} respectively. The group G is the fundamental group of a graph Γ of groups obtained from Δ by adding one edge e with associated group A . One end of e is at v_{13} and the other end has associated group D .

We take the splitting σ_1 of G to be the edge splitting of Γ associated to the edge e_1 . Thus σ_1 is a splitting of G of the form

$$G = \langle C_{13}, D \rangle *_{C_1} \langle C_{01}, C_{02}, C_{24} \rangle.$$

To describe the splitting σ_2 of G , we first slide the edge e so as to be attached to v_{24} instead of v_{13} . Now σ_2 is the edge splitting of this new graph of groups which is associated to the edge e_2 . Thus σ_2 is a splitting of G of the form

$$G = \langle C_{13}, C_{01}, C_{02} \rangle *_{C_2} \langle C_{24}, D \rangle.$$

It is easy to check that σ_1 and σ_2 are not compatible. If σ_1 and σ_2 cross strongly, then some conjugate of C_1 must intersect C_2 in a $VPC2$ subgroup. But our construction of G shows that conjugates of C_1 and C_2 intersect trivially or in a conjugate of A . It follows that σ_1 and σ_2 must cross weakly. Note that it also follows that C_1 and C_2 are not commensurable subgroups of G .

It remains to prove that the splittings σ_1 and σ_2 are 2-canonical. We will show that G has no nontrivial almost invariant subsets over any $VPC2$ subgroup, and that it has only one (up to equivalence, complementation and translation) nontrivial almost invariant subset over a $VPC1$ subgroup, which arises from the splitting τ of G over A associated to the edge e of Γ . Thus τ is the splitting of G as $G = H *_A D$. As each of σ_1 and σ_2 is clearly compatible with τ , it follows that they do not cross the associated almost invariant subset over A , and so are 2-canonical, as required.

First we prove the following technical result.

Lemma 3 *Let K be a subgroup of G such that, for any conjugate L of K in G ,*

1. the number $e(C_{ij}, L \cap C_{ij}) = 1$, for $ij = 13, 01, 02$ or 24 , and
2. the number $e(D, L \cap D) = 1$, and
3. $L \cap C_i$ has infinite index in C_i , for $i = 0, 1$ or 2 , and
4. $L \cap A$ has infinite index in A .

Then $e(G, K) = 1$.

Proof. Corresponding to the graph of groups decomposition Γ of G , there is a graph of spaces decomposition of a space X with fundamental group G . The covering space X_K of X with fundamental group K is a graph of spaces, where each vertex space has fundamental group equal to the intersection of K with a conjugate of some C_{ij} or of D , and each edge space has fundamental group equal to the intersection of K with a conjugate of C_0, C_1, C_2 or A . The hypotheses imply that each vertex space has 1 end and that each edge space is non-compact. It follows easily that X_K has 1 end, so that $e(G, K) = 1$ as required. ■

Now we summarise the main properties of G .

Lemma 4 *The group G has the following properties.*

1. G has 1 end.
2. If K is a $VPC1$ subgroup of G , then either $e(G, K) = 1$ or K is conjugate commensurable with A .
3. $Comm_G(A) = A$.
4. If K is a $VPC2$ subgroup of G , then $e(G, K) = 1$.

Proof. 1) Each C_{ij} is one-ended, as is D , and each of C_1, C_2 and A is infinite. Now we apply Lemma 3, with K equal to the trivial group, to see that $e(G) = 1$, as required.

2) Let K be a $VPC1$ subgroup of G . Thus the intersection of K with any subgroup of G is $VPCk$, for some $k \leq 1$. Now we apply Lemma 3. It follows that conditions 1)-3) of that lemma are satisfied. Hence $e(G, K) = 1$ unless condition 4) fails. This would mean that there is a conjugate L of K such that $L \cap A$ has finite index in A . As K and A are both $VPC1$, it follows that $e(G, K) = 1$ unless K is conjugate commensurable with A .

3) Let g denote an element of $Comm_G(A)$. As there is a homomorphism of G to \mathbb{Z} which maps A onto \mathbb{Z} , it follows that g must lie in $Z_G(kA)$, the centraliser of kA in G , for some non-zero k . Now recall that G is the fundamental group of the graph Γ of groups. As A is contained in each vertex group, and the centraliser in each vertex group of kA is equal to A , it follows that $Z_G(kA) = A$. Hence $Comm_G(A) = A$, as required.

4) Let K be a $VPC2$ subgroup of G . Thus the intersection of K with any subgroup of G is $VPCk$, for some $k \leq 2$. Now we apply Lemma 3. It follows that condition 1) and 3) of that lemma hold. As the intersection of any conjugate of K with D must be $VPCk$, with $k \leq 1$, it also follows that condition 2) holds. Hence $e(G, K) = 1$ unless condition 4) fails. This would mean that there is a conjugate L of K such that $L \cap A$ has finite index in A . As any $VPC2$ group has a subgroup of finite index isomorphic to $\mathbb{Z} \times \mathbb{Z}$, it follows that any $VPC1$ subgroup of K has large commensuriser in K . Thus this would imply that A has large commensuriser in G . Now part 3) of this lemma shows that this also is impossible. It follows that if K is a $VPC2$ subgroup of G , then $e(G, K) = 1$, as required. ■

Next we consider almost invariant subsets of G which are over a $VPC1$ subgroup.

Lemma 5 *G has only one (up to equivalence, complementation and translation) nontrivial almost invariant subset over a $VPC1$ subgroup, which arises from the splitting τ of G over A as $G = H *_A D$.*

Proof. Suppose that G has a nontrivial almost invariant subset over a $VPC1$ subgroup K , so that $e(G, K) > 1$. Recall from part 2) of Lemma 4 that if K is a $VPC1$ subgroup of G , then either $e(G, K) = 1$ or K is conjugate commensurable with A . As $Comm_G(A) = A$, by part 3) of Lemma 4, this implies that K is conjugate to a subgroup of A .

Now suppose that K is a subgroup of A and, as in the proof of Lemma 3, consider the cover X_K of X with fundamental group K . Part 3) of Lemma 4 tells us that $Comm_G(A) = A$, so that exactly one edge space of X_K is compact. Thus, as in the proof of Lemma 3, each of the two complementary components of this edge space has one end. It follows that $e(G, K) = 2$, for any subgroup K of finite index in A . Hence G has only one (up to equivalence, complementation and translation) nontrivial almost invariant subset over a $VPC1$ subgroup. As the splitting τ of G over A has such an almost invariant subset of G associated, the result follows. ■