In precalculus or calculus you may have studied conic sections with equations of the form

\[ Ax^2 + Cy^2 + Dx + Ey + F = 0 \]

Here we show that the general second-degree equation

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \]

can be analyzed by rotating the axes so as to eliminate the term \( Bxy \).

In Figure 1 the \( x \) and \( y \) axes have been rotated about the origin through an acute angle \( \theta \) to produce the \( X \) and \( Y \) axes. Thus, a given point \( P \) has coordinates \( (x, y) \) in the first coordinate system and \( (X, Y) \) in the new coordinate system. To see how \( X \) and \( Y \) are related to \( x \) and \( y \) we observe from Figure 2 that

\[
X = r \cos \phi \\
Y = r \sin \phi \\
x = r \cos(\theta + \phi) \\
y = r \sin(\theta + \phi)
\]

The addition formula for the cosine function then gives

\[
x = r \cos(\theta + \phi) = r(\cos \theta \cos \phi - \sin \theta \sin \phi) \\
= (r \cos \phi) \cos \theta - (r \sin \phi) \sin \theta = X \cos \theta - Y \sin \theta
\]

A similar computation gives \( y \) in terms of \( X \) and \( Y \) and so we have the following formulas:

\[
x = X \cos \theta - Y \sin \theta \\
y = X \sin \theta + Y \cos \theta
\]

By solving Equations 2 for \( X \) and \( Y \) we obtain

\[
X = x \cos \theta + y \sin \theta \\
Y = -x \sin \theta + y \cos \theta
\]
**EXAMPLE 1** If the axes are rotated through $60^\circ$, find the $XY$-coordinates of the point whose $xy$-coordinates are $(2, 6)$.

**SOLUTION** Using Equations 3 with $x = 2$, $y = 6$, and $\theta = 60^\circ$, we have

\[
X = 2 \cos 60^\circ + 6 \sin 60^\circ = 1 + 3\sqrt{3} \\
Y = -2 \sin 60^\circ + 6 \cos 60^\circ = -\sqrt{3} + 3
\]

The $XY$-coordinates are $(1 + 3\sqrt{3}, 3 - \sqrt{3})$.

Now let’s try to determine an angle $\theta$ such that the term $B_{xy}$ in Equation 1 disappears when the axes are rotated through the angle $\theta$. If we substitute from Equations 2 in Equation 1, we get

\[
A(X \cos \theta - Y \sin \theta)^2 + B(X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta) + C(X \sin \theta + Y \cos \theta)^2 + D(X \cos \theta - Y \sin \theta) + E(X \sin \theta + Y \cos \theta) + F = 0
\]

Expanding and collecting terms, we obtain an equation of the form

\[
A'X^2 + B'XY + C'Y^2 + D'X + E'Y + F = 0
\]

where the coefficient $B'$ of $XY$ is

\[
B' = 2(C - A) \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) = (C - A) \sin 2\theta + B \cos 2\theta
\]

To eliminate the $XY$ term we choose $\theta$ so that $B' = 0$, that is,

\[
(A - C) \sin 2\theta = B \cos 2\theta
\]

or

\[
\cot 2\theta = \frac{A - C}{B}
\]

**EXAMPLE 2** Show that the graph of the equation $xy = 1$ is a hyperbola.

**SOLUTION** Notice that the equation $xy = 1$ is in the form of Equation 1 where $A = 0$, $B = 1$, and $C = 0$. According to Equation 5, the $xy$ term will be eliminated if we choose $\theta$ so that

\[
\cot 2\theta = \frac{A - C}{B} = 0
\]
This will be true if \(2\theta = \pi/2\), that is, \(\theta = \pi/4\). Then \(\cos\theta = \sin\theta = 1/\sqrt{2}\) and Equations 2 become

\[
x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}} \quad y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}
\]

Substituting these expressions into the original equation gives

\[
\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right) = 1 \quad \text{or} \quad \frac{X^2}{2} - \frac{Y^2}{2} = 1
\]

We recognize this as a hyperbola with vertices \((\pm\sqrt{2}, 0)\) in the \(XY\)-coordinate system. The asymptotes are \(Y = \pm X\) in the \(XY\)-system, which correspond to the coordinate axes in the \(xy\)-system (see Figure 3).

**EXAMPLE 3** Identify and sketch the curve

\[73x^2 + 72xy + 52y^2 + 30x - 40y - 75 = 0\]

**SOLUTION** This equation is in the form of Equation 1 with \(A = 73\), \(B = 72\), and \(C = 52\). Thus

\[
cot 2\theta = \frac{A - C}{B} = \frac{73 - 52}{72} = \frac{7}{24}
\]

From the triangle in Figure 4 we see that

\[
\cos 2\theta = \frac{7}{24}
\]

The values of \(\cos\theta\) and \(\sin\theta\) can then be computed from the half-angle formulas:

\[
\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 + \frac{7}{24}}{2}} = \frac{4}{5}
\]

\[
\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1 - \frac{7}{24}}{2}} = \frac{3}{5}
\]

The rotation equations (2) become

\[
x = \frac{4}{5}X - \frac{3}{5}Y \quad y = \frac{1}{5}X + \frac{4}{5}Y
\]

Substituting into the given equation, we have

\[
73\left(\frac{4}{5}X - \frac{3}{5}Y\right)^2 + 72\left(\frac{4}{5}X - \frac{3}{5}Y\right)\left(\frac{1}{5}X + \frac{4}{5}Y\right) + 52\left(\frac{1}{5}X + \frac{4}{5}Y\right)^2
\]

\[+ 30\left(\frac{4}{5}X - \frac{3}{5}Y\right) - 40\left(\frac{1}{5}X + \frac{4}{5}Y\right) - 75 = 0
\]

which simplifies to

\[4X^2 + Y^2 - 2Y = 3\]

Completing the square gives

\[4X^2 + (Y - 1)^2 = 4 \quad \text{or} \quad X^2 + \frac{(Y - 1)^2}{4} = 1
\]

and we recognize this as being an ellipse whose center is \((0, 1)\) in \(XY\)-coordinates.
Since \( \theta = \cos^{-1}(\frac{3}{4}) \approx 37^\circ \), we can sketch the graph in Figure 5.

\[
\begin{align*}
73x^2 + 72xy + 52y^2 + 30x - 40y - 75 &= 0 \\
or \quad 4x^2 + (y-1)^2 &= 4
\end{align*}
\]

**FIGURE 5**

**Exercises**

**A Click here for answers.**

1-4 ■ Find the \( XY \)-coordinates of the given point if the axes are rotated through the specified angle.

1. (1, 4), 30°
2. (4, 3), 45°
3. (−2, 4), 60°
4. (1, 1), 15°

5-12 ■ Use rotation of axes to identify and sketch the curve.

5. \( x^2 - 2xy + y^2 - x - y = 0 \)
6. \( x^2 - xy + y^2 = 1 \)
7. \( x^2 + xy + y^2 = 1 \)
8. \( \sqrt{3}xy + y^2 = 1 \)
9. \( 97x^2 + 192xy + 153y^2 = 225 \)
10. \( 3x^2 - 12\sqrt{3}xy + 6y^2 + 9 = 0 \)
11. \( 2\sqrt{3}xy - 2y^2 - \sqrt{3}x - y = 0 \)
12. \( 16x^2 - 8\sqrt{2}xy + 2y^2 + (8\sqrt{2} - 3)x - (6\sqrt{2} + 4)y = 7 \)

13. (a) Use rotation of axes to show that the equation
    \( 36x^2 + 96xy + 64y^2 + 20x - 15y + 25 = 0 \)
    represents a parabola.

(b) Find the \( XY \)-coordinates of the focus. Then find the \( xy \)-coordinates of the focus.

(c) Find an equation of the directrix in the \( xy \)-coordinate system.

14. (a) Use rotation of axes to show that the equation
    \( 2x^2 - 72xy + 23y^2 - 80x - 60y = 125 \)
    represents a hyperbola.

(b) Find the \( XY \)-coordinates of the foci. Then find the \( xy \)-coordinates of the foci.

(c) Find the \( xy \)-coordinates of the vertices.

(d) Find the equations of the asymptotes in the \( xy \)-coordinate system.

(e) Find the eccentricity of the hyperbola.

15. Suppose that a rotation changes Equation 1 into Equation 4. Show that
    \[
    A' + C' = A + C
    \]

16. Suppose that a rotation changes Equation 1 into Equation 4. Show that
    \[
    (B')^2 - 4A'C' = B^2 - 4AC
    \]

17. Use Exercise 16 to show that Equation 1 represents (a) a parabola if \( B^2 - 4AC = 0 \), (b) an ellipse if \( B^2 - 4AC < 0 \), and (c) a hyperbola if \( B^2 - 4AC > 0 \), except in degenerate cases when it reduces to a point, a line, a pair of lines, or no graph at all.

18. Use Exercise 17 to determine the type of curve in Exercises 9–12.
**Answers**

1. \(((\sqrt{3} + 4)/2, (4\sqrt{3} - 1)/2)\)
2. \((2\sqrt{3} - 1, \sqrt{3} + 2)\)
3. \(X = \sqrt{2} Y^2\), parabola
4. \(X^2 + (Y^2/9) = 1\), ellipse
5. \(X = Y\), hyperbola
6. \(X^2 + Y^2 = 2\), ellipse

7. \(3X^2 + Y^2 = 2\), ellipse

8. \((X - 1)^2 - 3Y^2 = 1\), hyperbola

9. \((X - 1)^2 - 3Y^2 = 1\), hyperbola

10. \(X = Y\), hyperbola

11. \(X = Y\), hyperbola

12. \(X^2 + Y^2 = 2\), ellipse

13. (a) \(Y - 1 = 4X^2\)  
(b) \(0, \left(\frac{17}{16}\right), \left(-\frac{17}{20}, \frac{51}{50}\right)\)  
(c) \(64x - 48y + 75 = 0\)