velocity and acceleration vectors in terms of unit vectors \( \mathbf{t}, \mathbf{n}, \) and \( \mathbf{b} \) which are, respectively, tangent, normal, and binormal to the helix at \( P \).

**Answer:** The velocity vector is parallel to \( \mathbf{t} \) and has a magnitude \( v \). Hence \( v = \mathbf{v} \cdot \mathbf{t} \). By differentiation, and noting that \( v \) is a constant, we have \( \mathbf{d}v/\mathbf{d}t = v \mathbf{dt}/\mathbf{d}t \). But since \( \mathbf{t} \) has a constant length of unity, \( \mathbf{dt}/\mathbf{d}t \) must be perpendicular to \( \mathbf{t} \) and hence must be a combination of \( \mathbf{n} \) and \( \mathbf{b} \):

\[
\frac{\mathbf{dt}}{\mathbf{d}t} = \kappa \mathbf{n} + \tau \mathbf{b}
\]

where \( \kappa \) and \( \tau \) are constants. If the particle moves with unit velocity, the constants \( \kappa \) and \( \tau \) are called the curvature and the torsion of the space curve, respectively.

It is convenient to use polar coordinates for this problem. Let the unit vectors in the direction of the radial, circumferential, and axial directions be \( \mathbf{r}, \hat{\theta}, \) and \( \hat{z} \), respectively. Then

\[
\mathbf{v} = u \hat{\theta} + w \hat{z}
\]

where \( u \) and \( w \) are the circumferential and axial velocities, respectively. Hence

\[
\frac{dv}{dt} = (du/dt) \hat{\theta} + u \frac{d\hat{\theta}}{dt} + (dw/dt) \hat{z} + w \frac{d\hat{z}}{dt} = u \frac{d\hat{\theta}}{dt} = -(\kappa^2/a) \hat{r}.
\]

The velocities \( u \) and \( w \) are related to \( v \) as follows: In the time interval \( \Delta t = 2\pi/a \), the axial position \( z \) is changed by \( h \). Hence \( w = h/\Delta t = hu/2\pi a \), and

\[
v = u[1 + h^2/(4\pi^2 a^2)]^{1/2}.
\]

### 2.3 THE SUMMATION CONVENTION

For further development an important matter of notation must be mastered.

A set of \( n \) variables \( x_1, x_2, \ldots, x_n \) is usually denoted as \( x_i \), \( i = 1, \ldots, n \). When written singly, the symbol \( x_i \) stands for any one of the variables \( x_1, x_2, \ldots, x_n \). The range of \( i \) must be indicated in every case; the simplest way is to write, as illustrated here, \( i = 1, 2, \ldots, n \). The symbol \( i \) is an index. An index may be either a subscript or a superscript. A system of notations using indices is said to be an indicial notation.

Consider an equation describing a plane in a three-dimensional space referred to a rectangular Cartesian frame of reference with axes \( x_1, x_2, x_3 \),

\[
(2.3-1) \quad a_1 x_1 + a_2 x_2 + a_3 x_3 = p,
\]

where \( a_i \) and \( p \) are constants. This equation can be written as

\[
(2.3-2) \quad \sum_{i=1}^{3} a_i x_i = p.
\]
However, we shall introduce the summation convention and write the equation above in the simple form

\[ a_i x_i = p. \]

(2.3-3)

The convention is as follows: The repetition of an index in a term will denote a summation with respect to that index over its range. The range of an index \( i \) is the set of \( n \) integer values 1 to \( n \). An index that is summed over is called a dummy index; one that is not summed is called a free index.

Since a dummy index just indicates summation, it is immaterial which symbol is used. Thus, \( a_i x_i \) may be replaced by \( a_j x_j \), etc. This is analogous to the dummy variable in an integral

\[ \int_a^b f(x) \, dx = \int_a^b f(y) \, dy. \]

The use of the index and summation convention may be illustrated by other examples. Consider a unit vector \( \mathbf{v} \) in a three-dimensional Euclidean space with rectangular Cartesian coordinates \( x, y, \) and \( z \). Let the direction cosines \( \alpha_i \) be defined as

\[ \alpha_1 = \cos(\mathbf{v}, x), \quad \alpha_2 = \cos(\mathbf{v}, y), \quad \alpha_3 = \cos(\mathbf{v}, z), \]

where \((\mathbf{v}, x)\) denotes the angle between \( \mathbf{v} \) and the \( x \)-axis, and so forth. The set of numbers \( \alpha_i (i = 1, 2, 3) \) represents the components of the unit vector on the coordinates axes. The fact that the length of the vector is unity is expressed by the equation

\[ (\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 = 1, \]

or, simply,

(2.3-4)

\[ \alpha_i \alpha_i = 1. \]

As another illustration, consider a line element with components \( dx, dy, dz \) in a three-dimensional Euclidean space with rectangular Cartesian coordinates \( x, y, \) and \( z \). The square of the length of the line element is

(2.3-5)

\[ ds^2 = dx^2 + dy^2 + dz^2. \]

If we define

(2.3-6)

\[ dx_1 = dx, \quad dx_2 = dy, \quad dx_3 = dz, \]

Sec. 2.3 and

(2.3-7)

\[ \delta_{11} \]

(2.3-8) \( \delta_{12} \)

then (2.3-5) may be written

(2.3-9) \( \delta_{ij} \)

with the understanding that there are two summation symbols. The symbol \( \delta_{ij} \) as defined here is a matrix.

The following determinants

\[ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \]

If we denote the geneformal as \( |a_{ij}| \), then there

(2.3-10) \( e_{111} = e_{222} \)

where \( e_{ijk} \), the permu

(2.3-11) \( e_{123} = e_{311} \)

(2.3-12) \( e_{213} = e_{321} \)

In other words, \( e_{ijk} \) is 1 if the subquantities which will be by the identity

This \( e \cdot \delta \) identity is used it can be verified by

Finally, we shall formulas. Let \( f(x_1, \ldots) \)

Then its differential is

(2.3-12)

\[ df = \frac{\partial f}{\partial x_1} \]
and write the
\[
\delta_{11} = \delta_{22} = \delta_{33} = 1,
\]
\[
\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0,
\]
(2.3-7)

then (2.3-5) may be written as
\[
ds^2 = \delta_{ij} \, dx_i \, dx_j,
\]
(2.3-8)

with the understanding that the range of the indices \(i\) and \(j\) is 1 to 3. Note that there are two summations in the expression above, one over \(i\) and one over \(j\).

The symbol \(\delta_{ij}\) as defined in (2.3-7) is called the Kronecker delta.

The following determinant illustrates another application:

\[
\begin{vmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{23}a_{32} - a_{21}a_{33}a_{12} - a_{31}a_{12}a_{23}.
\]

If we denote the general term in the determinant by \(a_{ij}\) and write the determinant as \(|a_{ij}|\), then the equation above can be written as
\[
|a_{ij}| = e_{rns}a_{ri}a_{sj}a_{tk},
\]
(2.3-9)

where \(e_{rns}\), the permutation symbol, is defined by the equations
\[
e_{111} = e_{222} = e_{333} = e_{112} = e_{221} = e_{211} = e_{331} = \ldots = 0,
\]
(2.3-10)

\[
e_{123} = e_{231} = e_{312} = 1,
\]

\[
e_{213} = e_{123} = e_{132} = -1.
\]

In other words, \(e_{ijk}\) vanishes whenever the values of any two indices coincide; \(e_{ijk} = 1\) when the subscripts permute like 1, 2, 3; and \(e_{ijk} = -1\) otherwise.

The Kronecker delta and the permutation symbol are very important quantities which will appear again and again in this book. They are connected by the identity
\[
e_{ijk}e_{rst} = \delta_{jr}\delta_{sk} - \delta_{js}\delta_{kr},
\]
(2.3-11)

This \(e-\delta\) identity is used frequently enough to warrant special attention here. It can be verified by actual trial.

Finally, we shall extend the summation convention to differentiation formulas. Let \(f(x_1, x_2, \ldots, x_n)\) be a function of \(n\) variables \(x_1, x_2, \ldots, x_n\). Then its differential shall be written as
\[
df = \frac{\partial f}{\partial x_1} \, dx_1 + \frac{\partial f}{\partial x_2} \, dx_2 + \cdots + \frac{\partial f}{\partial x_n} \, dx_n = \frac{\partial f}{\partial x_i} \, dx_i.
\]
(2.3-12)
PROBLEMS

2.15 Write Eq. (2.2-1) or (2.2-3) in the index form. Let the components of \( F^{(i)} \) be written as \( F^i_k \), \( k = 1, 2, 3 \); i.e., \( F_x = F_1 \), etc.

*Answer:* \( \sum_{i=1}^{3} F^i_k = 0 \).

2.16 Show that
(a) \( \delta_{ii} = 3 \)
(b) \( \delta_{ij} \delta_{ij} = 3 \)
(c) \( e_{ijk} e_{ijl} = 6 \)
(d) \( e_{ijk} A_j A_k = 0 \)
(e) \( \delta_{ij} \delta_{jk} = \delta_{ik} \)
(f) \( \delta_{ij} \delta_{ik} = 0 \)

2.17 Write Eqs. (2.1-1), (2.1-5) in the index form, e.g., \( u \cdot v = u^i v_i \).

*Note:* For Eq. (2.1-1), we may do the following: Define three unit vectors \( v^{(i)} = i, v^{(j)} = j, v^{(k)} = k \); then \( u = u^i v_i \).

*Answer:* \( u = u^i v_i \), \( u \cdot v = u^i v_i \).

2.18 Use the index form of vector equations to solve Prob. 2.5 through 2.9.

2.19 The vector product of two vectors \( \mathbf{u} = (u_1, u_2, u_3) \) and \( \mathbf{v} = (v_1, v_2, v_3) \) is the vector \( \mathbf{w} = \mathbf{u} \times \mathbf{v} \) whose components are

\[
 w_1 = u_2 v_3 - u_3 v_2, \quad w_2 = u_3 v_1 - u_1 v_3, \quad w_3 = u_1 v_2 - u_2 v_1.
\]

Show that this can be shortened by writing

\[
 w_i = e_{ijk} u_j v_k.
\]

2.20 Express Eq. (2.1-7) in the index form.

2.21 Derive the vector identity connecting three arbitrary vectors \( \mathbf{A}, \mathbf{B}, \mathbf{C} \) by the method of vector analysis:

\[
 \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.
\]

*Solution:* Since \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \) is perpendicular to \( \mathbf{B} \times \mathbf{C} \), it must lie in the plane of \( \mathbf{B} \) and \( \mathbf{C} \). Hence we may write \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = a\mathbf{B} + b\mathbf{C} \) where \( a, b \) are scalar quantities. But \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \) is a linear function of \( \mathbf{A}, \mathbf{B}, \) and \( \mathbf{C} \); hence \( a \) must be a linear scalar combination of \( \mathbf{A} \) and \( \mathbf{C} \), \( b \) must be one of \( \mathbf{A} \) and \( \mathbf{B} \). Hence \( a, b \) are proportional to \( \mathbf{A} \cdot \mathbf{C} \) and \( \mathbf{A} \cdot \mathbf{B} \), respectively, and we may write

\[
 \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \lambda(\mathbf{A} \cdot \mathbf{C})\mathbf{B} + \mu(\mathbf{A} \cdot \mathbf{B})\mathbf{C}
\]

where \( \lambda, \mu \) are pure numbers, independent of \( \mathbf{A}, \mathbf{B}, \) and \( \mathbf{C} \). We can, therefore, evaluate \( \lambda, \mu \) by special cases, e.g., if \( i, j, k \) are the unit vectors in the directions of \( x-, y-, z \)-axes (right-handed rectangular Cartesian), respectively, we may put \( \mathbf{B} = i, \mathbf{C} = j \) to show that \( \lambda = 1 \).

2.22 Write the equation by means of the \( \varepsilon_{ijk} \) identity.

*Note:* Since the \( \mathbf{B}, \mathbf{C} \), this verification is a translation.

*Solution:* \( [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] \).

2.4 TRANSLATION AND

Consider two sets of \( O'-x'y'z' \) on a plane. If there is a shift of origin with a translation. If a point in the old and new frames of

\[
(2.4-1) \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

If the origin remains, \( Ox \) and \( Oy \) through an \( x \) transformation of axes relative to the old and new frames.

\[
(2.4-2) \quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}
\]
may put \( \mathbf{B} = i, \mathbf{C} = j, \mathbf{A} = 1 \) to show that \( \lambda = -1 \); and \( \mathbf{B} = i, \mathbf{C} = j, \mathbf{A} = j \) to show that \( \lambda = 1 \).

2.22 Write the equation in Prob. 2.21 in the index form, and prove its validity by means of the \( e \cdot \delta \) identity (2.3-11).

Note: Since the equation in Prob. 2.21 is valid for arbitrary vectors \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \) this verification may be regarded as a proof for the \( e \cdot \delta \) identity.

Solution: \( [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_i = e_{ijk}a_k(b \times c)_j = e_{ijk}a_ka_jb_i. \)

By the \( e \cdot \delta \) identity, Eq. (2.3-11), this becomes \( \delta_{ij}b_k - \delta_{kj}a_jb_i. \)

Hence it is \( \delta_{ij}a_m b_j - \delta_{jk}a_m b_i = a_m b_j - a_m b_i = (A \cdot B)_i - (A \cdot B)_j. \)

2.4 TRANSLATION AND ROTATION OF COORDINATES

Consider two sets of rectangular Cartesian frames of reference \( O-xy \) and \( O'-x'y' \) on a plane. If the frame of reference \( O'-x'y' \) is obtained from \( O-xy \) by a shift of origin without a change in orientation, then the transformation is a translation. If a point \( P \) has coordinates \( (x, y) \) and \( (x', y') \) with respect to the old and new frames of reference, respectively, then

\[
\begin{align*}
(x, y) & \quad \text{or} \quad (x', y') \\
(2.4-1) \\
& \begin{cases}
x' = x + h \\
y' = y + k
\end{cases}
\]

If the origin remains fixed, and the new axes are obtained by rotating \( Ox \) and \( Oy \) through an angle \( \theta \) in the counterclockwise direction, then the transformation of axes is a rotation. Let \( P \) have coordinates \( (x, y), (x', y') \) relative to the old and new frames of reference, respectively. Then (see Fig. 2.2),

\[
\begin{align*}
x &= x' \cos \theta - y' \sin \theta \\
y &= x' \sin \theta + y' \cos \theta.
\end{align*}
\]

Fig. 2.2 Rotation of coordinates.