Synchronization and relaxation for a class of globally coupled Hamiltonian systems

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Received 6 May 1997; received in revised form 12 April 1998; accepted 6 May 1998
Communicated by J.D. Meiss

Abstract

A class of coupled Hamiltonian systems is examined in which identical nonlinear oscillators are coupled through a mean field. The system is shown to have a steady desynchronized solution which becomes linearly unstable as the coupling strength is increased. We observe, in the stable case that the order parameter of the system decays to zero. For a wide class of initial conditions, the decay is exponential on an intermediate time scale and then as \( t^{-3/2} \), as \( t \to \infty \). This system shares many similarities to the Vlasov–Poisson equation and as well as Kuramoto’s model. © 1998 Elsevier Science B.V.

1. Introduction

This paper examines the behavior of a collection of identical nonlinear oscillators coupled through a mean field. Each oscillator is a single degree of freedom Hamiltonian system and the coupling is chosen so that the resulting coupled system will also be Hamiltonian. We shall see that such a system can be realized in physical settings. The resulting system is a Hamiltonian version of Kuramoto’s model [1,2] which describes a population of phased coupled oscillators. Kuramoto’s model is

\[
\theta_k = \omega_k + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_k) \quad \text{for} \quad k = 1, 2, \ldots, N,
\]

where \( N \gg 1 \) and \( \omega_k \) is sampled from a probability distribution, \( g(\omega) \). The coupling term in Kuramoto’s model is an example of a mean field. The aim of this model is to understand the dynamics of weakly coupled oscillators with attracting limit cycles. In Kuramoto’s model the frequency of the \( k \)th limit cycle is \( \omega_k \). Thus, Kuramoto’s model describes the dynamics of a collection of different oscillators coupled through a mean field. There have been several extensions and generalizations of Kuramoto’s model [5–10].
The phase coherence is measured by the complex order parameter,

\[ r e^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}. \]

When \( r = 0 \), the oscillators are desynchronized; this is called the incoherent or desynchronized state. Kuramoto showed that, as \( N \to \infty \), the desynchronized state is a steady solution. Furthermore, he showed that a transition occurs at \( K = K_c = 2/\pi g(0) \). For \( K > K_c \) the desynchronized state is unstable, often a supercritical bifurcation takes place (under some circumstances the bifurcation is subcritical, see [4] for more details).

For \( K < K_c \), Strogatz et al. [3] show that the order parameter, \( r(t) \), will tend to zero for small perturbations. They show that the nature of the decay depends both on the initial conditions and the distribution of the oscillator frequencies. They took \( N \to \infty \) and derived a kinetic equation to describe the probability density of the oscillators in phase space.

The reason that \( \lim_{t \to \infty} r(t) = 0 \) can be understood intuitively from the \( K = 0 \) case. Since the \( \omega_k \)'s are all incommensurate, one would expect as \( t \to \infty \) the \( \theta_k \)'s would become equally distributed on the interval \([0, 2\pi]\), resulting in \( r(t) \to 0 \). This can be explained more concretely from [3]; it follows from Eq. (5) of [3] when \( K = 0 \) that

\[ r(t) = \int_{-\infty}^{\infty} c_0(\omega)g(\omega)e^{-i\omega t} d\omega \]  

(1)

where \( c_0(\omega) \) is determined from the initial conditions. It follows that \( \lim_{t \to \infty} r(t) = 0 \) if \( c_0(\omega)g(\omega) \in L^1(\mathbb{R}) \) by the Riemann–Lebesgue lemma. It is shown in [3] that such behavior persists as long as \( K < K_c \).

It also follows from (1) that if all the oscillators are identical then \( \lim_{t \to \infty} r(t) \neq 0 \). This is because if the oscillators are identical then they all must have the same frequency, say \( \omega_s \), thus \( g(\omega) = \delta(\omega - \omega_s) \) (here \( \delta \) denotes the Dirac delta function). In this case we find

\[ r(t) = c_0(\omega_s)e^{-i\omega_st}; \]

clearly, in this case \( r(t) \) does not tend to zero. Thus, we see that a distribution of frequencies is crucial for damping to occur.

The procedure followed in [3] is similar in many respects to that used by Landau [11] in his study of the decay of the electric field in a collisionless plasma, modeled by the Vlasov–Poisson equation. The Vlasov–Poisson equation is a Hamiltonian system where the charged particles are coupled through their electric field. In the zero coupling limit each particle moves with constant speed (called streaming). The Vlasov–Poisson equation has an “incoherent state” corresponding to a spatially uniform plasma. Landau showed that the Fourier transform of the perturbed electric field can decay exponentially fast; this is now called Landau damping. The electric field is analogous to the order parameter in Kuramoto’s model. A similar phenomenon was found in a model for bubbly fluids [13].

Landau assumed that the background velocity distribution and the perturbation could be analytically continued as entire functions. The assumption of the perturbation as an entire function was criticized as being unphysical by Penrose [12] and Weitzner [14]. Nevertheless they both showed that the electric field will, under certain conditions, damp to zero. Weitzner [14] proved that the electric field could decay to zero with almost any time behavior depending on the initial data. Weitzner [15] and Weitzner and Dobrott [16] demonstrated that on an intermediate time scale there would be exponential decay. Similar results were reported in [3] for Kuramoto’s model.
It is natural to ask if similar behavior will also occur for globally coupled nonlinear oscillators where the uncoupled system is a single degree of freedom Hamiltonian. The following system is motivated by Kuramoto’s model:

\[
\begin{align*}
\dot{q}_k &= \frac{\partial}{\partial p_k} h(q_k, p_k) + \frac{\varepsilon}{2\pi N} \frac{\partial}{\partial p_k} f(q_k, p_k) \sum_{j=1}^{N} f(q_j, p_j), \\
\dot{p}_k &= -\frac{\partial}{\partial q_k} h(q_k, p_k) - \frac{\varepsilon}{2\pi N} \frac{\partial}{\partial q_k} f(q_k, p_k) \sum_{j=1}^{N} f(q_j, p_j),
\end{align*}
\]

(2)

where \( h(q, p) \) is the single-particle Hamiltonian, \( f(q, p) \) is the coupling term, and \( \varepsilon/2\pi \) is the coupling strength. The coupling was chosen so that (2) is Hamiltonian. The Hamiltonian for (2) is

\[
H = \sum_{i=1}^{N} h(q_i, p_i) + \frac{\varepsilon}{4\pi N} \sum_{i=1}^{N} \sum_{j=1}^{N} f(q_i, p_i) f(q_j, p_j).
\]

(3)

Here, we shall only consider \( h(q, p) \) such that all solutions of

\[
\dot{q} = \frac{\partial}{\partial p} h(q, p), \quad \dot{p} = -\frac{\partial}{\partial q} h(q, p)
\]

are periodic. Furthermore, as we shall see below, the above system admits a desynchronized state corresponding to the oscillators being out of phase with each other. Therefore we see that (2) is similar to Kuramoto’s model in that it has a collection of oscillators coupled through a mean field. In this sense (2) can be thought of as a Hamiltonian version of Kuramoto’s model.

Eq. (2) differs from Kuramoto’s model because all of the oscillators are identical. Even though the oscillators are identical we still expect a damping effect similar to that found in Kuramoto’s model because the period of the oscillation will depend on its amplitude. This means a distribution in amplitude will result in a distribution of oscillator frequencies resulting in damping.

Because of its connections with other well studied systems (2) is an interesting system worth studying. Moreover, there are physical systems that take the form given by (2). We will now discuss two such examples.

1.1. Electric circuit

Consider the electric circuit shown in Fig. 1. In this circuit we have \( N \) inductor–capacitor circuits coupled together by a single chain of inductors. In the above circuit all of the inductors have the same self-inductance denoted \( L \). In addition, all of the capacitors are identical, the voltage drop across the capacitor is denoted \( V(q) \) where \( q \) is the charge on the capacitor. We shall consider nonlinear capacitors so that \( V \) is not a linear function of \( q \). On the other hand the inductors will be given an air core and thus will be linear. In the \( k \)th inductor–capacitor circuit Kirchhoff’s law tells us that

\[
L \dot{J}_k + V(q_k) + M \dot{K} = 0,
\]

(4)

where \( J_k \) is the current in \( k \)th circuit, \( q_k \) is the charge on the \( k \)th capacitor, \( K \) is the current in the coupling circuit, and \( M \) is the mutual inductance between the inductor in the \( k \)th circuit and the coupling circuit. In the coupling circuit Kirchhoff’s law demands

\[
NL \ddot{K} + M \sum_{k=1}^{N} \dot{J}_k = 0.
\]

(5)
We use (5) to eliminate $K$ in (4) and obtain

$$LJ_k + V(q_k) = \frac{M^2}{NL} \sum_{k=1}^{N} J_k.$$  \hspace{1cm} (6)

It is useful to consider the following change of variables:

$$p_k = LJ_k - \frac{M^2}{NL} \sum_{k=1}^{N} J_k,$$

then (6) becomes

$$\dot{p}_k = -V(q_k).$$  \hspace{1cm} (7)

Next, we solve for $J_k$ in terms of $p_k$ to find

$$J_k = p_k - \frac{\varepsilon}{N} \sum_{i=1}^{N} p_i,$$

where $\varepsilon = M^2/(L^2 - M^2)$. Since $q_k = J_k$ it follows

$$\dot{q}_k = p_k - \frac{\varepsilon}{N} \sum_{i=1}^{N} p_i.$$  \hspace{1cm} (8)

Therefore we see that the equations of motion for this network are given by (7) and (8). These equations of motion are a Hamiltonian system with

$$H = \sum_{i=1}^{N} \left( \frac{1}{2} p_i^2 + U(q_i) \right) + \frac{\varepsilon}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} p_i p_j,$$  \hspace{1cm} (9)

where $U'(q) = V(q)$, which is of the same form as given by (3). It is also possible to construct mechanical systems having a Hamiltonian similar to (9).
1.2. Bubbly fluids

The propagation of sound through an ideal bubbly fluid can be understood by using the assumption that bubbles only change their radii and not their position. The sound propagation is governed by bubble–bubble interactions. A single bubble containing an ideal gas surrounded by an ideal fluid is a nonlinear oscillator. This has been shown in [17–19], for example. Effective equations were derived in [17,18] describing sound propagation in a bubbly fluid. In a generalization of this work on bubbly fluids the author [20] has derived a system of equations which takes a form very similar to that of (2). This set of equations is studied using the ideas developed in this paper. It is observed that there is a damping mechanism for the propagation of sound in a bubbly fluid very much like Landau damping.

1.3. Outline

The outline of this paper is as follows: we will present a kinetic equation describing (2) as \( N \to \infty \) and identify the desynchronized state. The stability of the desynchronized state is studied by first transforming equations of motion to action–angle variables. We then linearize the equations of motion about the desynchronized state. The linearized equation is studied using techniques based on the Laplace transform, similar to those used it [3,11,14]. This allows us to obtain estimates on the decay of the order parameter and find a condition for linear stability. We show that the order parameter can relax back to zero exponentially fast on an intermediate time scale, in a fashion similar to the Kuramoto model and the Vlasov–Poisson system. We find, however, that in the limit \( t \to \infty \), the natural order parameter, for this problem, decays like \( t^{-3/2} \) as \( t \to \infty \) for a large class of initial data. This is in contrast to the Kuramoto model and Vlasov–Poisson system where the long time decay depends sensitively on the initial conditions. We conclude with an example in which a numerical solution is compared with the asymptotic expressions.

2. Coupled Hamiltonian system

Motivated by the discussion in the Section 1 we will consider Hamiltonians of the form:

\[
H = \sum_{i=1}^{N} h(q_i, p_i) + \frac{\epsilon}{4\pi N} \sum_{i=1}^{N} \sum_{j=1}^{N} f(q_i, p_i) f(q_j, p_j),
\]

(10)

where \( h \) is the single-particle Hamiltonian and \( \epsilon \) is the coupling parameter. As mentioned in Section 1, the single-particle Hamiltonian has a single degree of freedom such that all of its orbits are periodic. This means that all the level curves of \( h(q, p) \) are closed curves.

2.1. Kinetic equation

In this section we shall derive a kinetic equation for the density of oscillators in phase space, \((q, p)\), as \( N \to \infty \). Our starting point will be the equations of motion:

\[
\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}.
\]

For finite \( N \), the number density in phase-space is given by

\[
\rho_N(q, p, t) = \frac{1}{N} \sum_{k=1}^{N} \delta(q - q_k(t)) \delta(p - p_k(t)).
\]
Next we let $N \to \infty$ and define $\rho = \lim_{N \to \infty} \rho_N$. We then find that $\rho(q, p, t)$ satisfies the kinetic equation

$$\frac{\partial \rho}{\partial t} + U \frac{\partial \rho}{\partial q} + F \frac{\partial \rho}{\partial p} = 0,$$

where

$$U = \frac{\partial h}{\partial p} + \varepsilon \frac{\partial f}{\partial p} j(t), \quad F = -\frac{\partial h}{\partial q} - \varepsilon \frac{\partial f}{\partial q} j(t),$$

and

$$j(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(q, p) \rho(q, p, t) \, dp \, dq.$$

This equation can be derived using the Klimontovich equation for $\rho_N(p, q, t)$, and taking the limit $N \to \infty$ (see, for example, [23–25]).

We see that $\rho = \rho_0(h(p, q))$ is a steady solution of the above kinetic equation provided that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(q, p) \rho_0(h(q, p)) \, dp \, dq = 0.$$  

This will be true in many cases for any $\rho_0$; for example, $h(p, q)$ is an even function of $p$ and $f(p, q)$ is an odd function of $p$. This steady solution corresponds to a situation where the oscillators are out of phase or desynchronized. The goal of this paper is to study this desynchronized state, which is most easily done in action–angle coordinates.

2.2. Action–angle variables

It is convenient to write the Hamiltonian in terms of the action–angle coordinates of the single-particle Hamiltonian which we write as

$$q = Q(\theta, I) \quad \text{and} \quad p = P(\theta, I).$$

In these variables, the Hamiltonian is

$$H = \sum_{i=1}^{N} h(I_i) + \varepsilon \frac{1}{4\pi N} \sum_{i=1}^{N} \sum_{j=1}^{N} f(\theta_i, I_i) f(\theta_j, I_j).$$

by $h(I)$ and $f(\theta, I)$ we mean $h(Q(\theta, I), P(\theta, I))$ and $f(Q(\theta, I), P(\theta, I))$.

2.2.1. Assumptions

The frequency of the single-particle is $h'(I) = \omega(I)$. We shall only consider systems where $\omega(I) > 0$, and the behavior for small amplitude motion is simple harmonic motion with frequency $\omega_0$. We shall assume that $\omega(I)$ is strictly monotonic; $\omega'(I) \neq 0$ for all $I \geq 0$. This is equivalent to having the Hamiltonian to be taken as a strictly convex function of the action, $I$. This will guarantee that all the solutions of the single-particle Hamiltonian are periodic.

The coupling function, $f(\theta, I)$, must satisfy the following conditions:

$$\int_{0}^{2\pi} f(\theta, I) \, d\theta = 0$$
and
\[ f(\theta, I) = O(\sqrt{I}) \quad \text{as } I \to 0. \]  
(15)

Condition (14) implies that (12) and (15) combined with the assumptions about \( h(I) \) ensure that \( f(p, q) = O(p) + O(q) \) as \( q \to 0 \) and \( p \to 0 \) (hence for small amplitude oscillations the coupling is linear). We will also assume that \( f(\theta, I) \) is \( C^\infty \) for all \( \theta \) and all \( I > 0 \).

2.2.2. Kinetic equation

In this section we shall derive a kinetic equation for the density of oscillators in the action–angle phase space, \((\theta, I)\), as \( N \to \infty \). The starting point is the equations of motion in action–angle variables,
\[ \dot{\theta}_k = \frac{\partial H}{\partial I_k} \quad \text{and} \quad \dot{I}_k = -\frac{\partial H}{\partial \theta_k}. \]

We find
\[ \dot{I}_k = -\frac{\varepsilon}{2\pi N} \frac{\partial f(\theta_k, I_k)}{\partial \theta_k} \sum_j f(\theta_j, I_j), \quad \dot{\theta}_k = \omega(I_k) + \frac{\varepsilon}{2\pi N} \frac{\partial f(\theta_k, I_k)}{\partial I_k} \sum_j f(\theta_j, I_j), \]
where \( \omega(I) = h'(I) \). For finite \( N \) the number density in phase space is given by
\[ \rho_N(\theta, I, t) = \frac{1}{N} \sum_{k=1}^{N} \delta(\theta - \theta_k(t))\delta(I - I_k(t)). \]

Next, we let \( N \to \infty \), and define \( \rho = \lim_{N \to \infty} \rho_N \). It is then found that \( \rho(\theta, I, t) \) satisfies the kinetic equation
\[ \frac{\partial \rho}{\partial t} + \Omega \frac{\partial \rho}{\partial \theta} + F \frac{\partial \rho}{\partial I} = 0, \]  
(16)

where
\[ \Omega = \omega(I) + \varepsilon \frac{\partial f}{\partial I} j(t), \quad F = -\varepsilon \frac{\partial f}{\partial \theta} j(t), \]
and
\[ j(t) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty f(\theta, I) \rho(\theta, I, t) \, dI \, d\theta. \]

2.3. Desynchronized solutions

If we consider \( \rho(\theta, I, t) = \rho_0(I) \), it is clear that (14) implies \( j(t) = 0 \). This implies that \( \rho_0(I) \) is a steady solution of (16). This is of course the same steady solution discussed above. Such a solution corresponds to the situation where the oscillators are desynchronized, or in other words, uniformly distributed in phase. It is clear that \( j(t) \) plays the role of the order parameter in this problem; when \( j(t) = 0 \) we then have a steady desynchronized solution.

We make the following assumptions about the desynchronized state:
\[ \rho_0(I) \in C^3 \quad \text{and} \quad \lim_{I \to \infty} \rho_0(I) = 0. \]  
(17)
Furthermore, we assume that $\rho_0'(I)$ tends to zero fast enough, so that the following conditions are satisfied:

$$\int_0^\infty f^2(\theta, I)\rho_0'(I)\,dI < \infty \quad \text{if } \omega'(I) > 0$$

(18)

and

$$\int_0^\infty \frac{f^2(\theta, I)\rho_0'(I)}{\omega^N(I)}\,dI < \infty \quad \text{for } N = 1, 2, 3\ldots \quad \text{if } \omega'(I) < 0.$$  

(19)

At the present time we do not have a physical interpretation of (18) or (19), but they do not seem to impose any great restriction; for example, any $\rho_0(I)$ with compact support would suffice.

3. Linearized dynamics

Here we examine the linear stability of the desynchronized state. We let

$$\rho(\theta, I, t) = \rho_0(I) + g(\theta, I, t) \quad g \ll 1.$$  

(20)

We shall make the assumptions about the initial perturbation, $g(\theta, I, 0)$:

$$g(\theta, I, 0) \in C^2, \quad \lim_{I \to -\infty} g(\theta, I, 0) = 0,$$

(21)

$$\int_0^{\infty} f(\theta, I)g(\theta, I, 0)\,dI < \infty \quad \text{if } \omega'(I) > 0,$$

(22)

and

$$\int_0^{\infty} \frac{f(\theta, I)g(\theta, I, 0)}{\omega^N(I)}\,dI < \infty \quad \text{for } N = 1, 2, 3, \ldots \quad \text{if } \omega'(I) < 0.$$  

(23)

Remark. Assumptions (17) – (23) are used to insure that the integrals and sums which arise later will converge; these assumptions can probably be weakened somewhat.

Eq. (20) is substituted into (16), the nonlinear terms in $g$ are dropped and we obtain the following equation:

$$\frac{\partial g}{\partial t} + \omega(I)\frac{\partial g}{\partial \theta} - \varepsilon \frac{\partial f}{\partial \theta} j(t)\rho_0'(I) = 0,$$

(24)

where

$$j(t) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} f(\theta, I)g(\theta, I, t)\,dI\,d\theta.$$  

(25)

Next, we let

$$g(\theta, I, t) = \sum_{m=-\infty}^{\infty} g_m(I, t)e^{im\theta} \quad \text{and} \quad f(\theta, I) = \sum_{m=-\infty}^{\infty} f_m(I)e^{im\theta}.$$
Therefore (24) becomes
\[ \frac{\partial g_m}{\partial t} + im\omega(I)g_m - \varepsilon imf_m(I)\rho'_0(I)j(t) = 0, \]  
(26)

where
\[ j(t) = \sum_{m=-\infty}^{\infty} \int_0^\infty g_{-m}(I, t)f_m(I)\,dI. \]

Since the small amplitude behavior is simple harmonic motion in the uncoupled case and the coupling is linear, we know
\[ f_m(I) \sim c_m |m|^{1/2} \quad \text{as} \; I \to 0, \]  
(27)

where \( c_m \) is a constant with \( c_1 \neq 0 \). We note that (14) implies that \( f_0(I) = 0 \). Next we take the Laplace transform of (26) to obtain
\[ G_m(I, s) = \frac{g_m(I, 0)}{s + im\omega(I)} + \frac{\varepsilon imf_m\rho'_0(I)}{s + im\omega(I)}J(s), \]

where
\[ G_m(I, s) = \int_0^\infty e^{-st}g_m(I, t)\,dt \quad \text{and} \quad J(s) = \int_0^\infty e^{-st}j(t)\,dt. \]

We multiply by \( f_{-m} \), then sum over \( m \), integrate over \( I \) and solve for \( J(s) \). We then find
\[ J(s) = \frac{A(s)}{B(s)}, \]

where
\[ A(s) = \sum_{k=1}^{\infty} \int_0^\infty \left( \frac{f_{-k}(I)g_k(I, 0)}{s + ik\omega(I)} + \frac{f_k(I)g_{-k}(I, 0)}{s - ik\omega(I)} \right)\,dI \]

and
\[ B(s) = 1 - 2\varepsilon \sum_{k=1}^{\infty} \int_0^\infty \frac{k^2 f_k(I)f_{-k}(I)\rho'_0(I)\omega(I)}{s^2 + k^2\omega^2(I)}\,dI. \]

In the above formulas we have used \( f_0(I) = 0 \). In the following analysis we shall always take \( k \) to be a positive integer.

To determine \( j(t) \), we take the inverse Laplace transform,
\[ j(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} J(s)e^{st}\,ds, \]
where \( \gamma \) is sufficiently large so that the path of integration is to the right of any singularity of the integrand. It is convenient to use the variable, \( s = -iz \); thus,

\[
j(t) = \frac{1}{2\pi i} \int_{-\infty + i\gamma}^{\infty + i\gamma} \frac{T(z)}{D(z)} e^{-izt} \, dz.
\]

where

\[
T(z) = \sum_{k=1}^{\infty} T_k(z) \quad \text{with} \quad T_k(z) = \int_0^\infty \left( \frac{f_{-k}(I)g_k(I,0)}{k\omega(I) - z} - \frac{f_k(I)g_{-k}(I,0)}{k\omega(I) + z} \right) \, dI
\]

and

\[
D(z) = 1 - \varepsilon \sum_{k=1}^{\infty} D_k(z) \quad \text{with} \quad D_k(z) = \int_0^\infty \frac{2k^2 f_k(I) f_{-k}(I) \rho_0(I) \omega(I) \, dI}{k^2 \omega^2(I) - z^2}.
\]

Now the path of integration is above any singularities of the integrand.

With \( j(t) \) known we can readily solve for \( g_k(I, t) \):

\[
g_k(I, t) = g_k(I, 0)e^{-i\omega(I)t} + i\varepsilon k f_k(I) \rho_0(I) \int_0^t e^{i\omega(I)(t-\tau)} j(\tau) \, d\tau.
\]

In this form it is evident that the collective effects are contained in \( j(t) \). Furthermore, one can use the above result to obtain the following integral equation for \( j(t) \):

\[
j(t) = \sum_{m=-\infty}^{\infty} \int_0^\infty f_{-m}(I) g_m(I, 0) e^{-im\omega(I)t} \, dI
\]

\[
+ 2\varepsilon \sum_{k=1}^{\infty} \int_0^\infty \int_0^t f_k(I) f_{-k}(I) \sin [k\omega(I)(t-\tau)] j(\tau) \, dI \, d\tau.
\]

(29)

3.1. Collective behavior

The goal of this section will be to understand the behavior of \( j(t) \). It is a simple matter to verify, using the assumptions we have made, that both \( T(z) \) and \( D(z) \) are analytic functions, provided that \( \text{Im}(z) \neq 0 \). We will now assume that \( D(z) \) has only simple zeros. It then follows that the integrand of (28) is meromorphic, with poles located at the zeros of \( D(z) \). Since \( D(\bar{z}) = \overline{D(z)} \) it follows that if \( z \) is a zero then so is \( \bar{z} \). Further, \(-\bar{z}\) and \(-z\) are also zeros since \( D(z) = D(-z) \).

Claim. For \( t > 0 \)

\[
\int_{-\infty - i\gamma}^{\infty - i\gamma} \frac{T(z)}{D(z)} e^{-izt} \, dz = 0.
\]

Proof. In view of the above remarks the integrand is analytic for \( \text{Im}(z) < -\gamma \). Since \( T(z) \rightarrow 0 \) and \( D(z) \rightarrow 1 \) as \( |z| \rightarrow \infty \), we may use Jordan's lemma to prove the above claim. \( \square \)
In view of the above claim, (28) can be rewritten as

\[ j(t) = \frac{1}{2\pi i} \left[ \int_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \right] \frac{T(z)}{D(z)} e^{-izt} \, dz. \]

Next, we take \( \gamma \to 0 \), while deforming the contour around the zeros of \( D(z) \). Let \( z_m, \, m = 1 \) to \( M \), denote the roots of \( D(z) = 0 \), with \( \text{Im}(z_m) \neq 0 \), then

\[ j(t) = -\sum_{m=1}^{M} \frac{T(z_m)}{D'(z_m)} e^{-iz_m t} + \frac{1}{2\pi i} \left[ \int_{-\infty}^{\infty} \frac{T(z)}{D(z)} \right]_f e^{-izt} \, dx, \quad (30) \]

where \([\cdot]_f\) denotes the jump across the \( x \)-axis. This construction was used by Weitzer [14] and Crawford and Hislop [21]. It is evident therefore, that if \( D(z) \) has any zeros with \( \text{Im}(z) \neq 0 \), then \( j(t) \) must grow exponentially in time; if \( D(z) \) has only zeros with \( \text{Im}(z) = 0 \), then \( j(t) \) will oscillate as \( t \to \infty \). On the other hand, if \( D(z) \) does not have any zeros we shall show that for reasonable initial conditions, \( j(t) \to 0 \) as \( t \to \infty \).

**Remark.** \( D(z) \) can only have a finite number of zeros with \( \text{Im}(z) \neq 0 \). This follows because the zeros of \( D(z) \) are isolated and \( D(z) \to 1 \) as \( |z| \to \infty \) (the zeros of analytic function are isolated).

### 3.1.1. Relaxation to the desynchronized state

The goal of this section is to show that \( j(t) \to 0 \) when \( D(z) \) has no zeros. We shall use the form of \( j(t) \) given by (30) for this purpose. Therefore, it is necessary to compute \( T(z) \) and \( D(z) \) on either side of the \( x \)-axis. A simple modification of the Plemelj formula will be used for this purpose. The modification of the Plemelj formula used here is

\[ \lim_{\gamma \to 0} \frac{1}{k\omega(I) \pm (x + iy)} = P \left( \frac{1}{k\omega(I) \pm x} \right) \mp \frac{i\pi \text{sgn}(y)H(\mp x)}{k|\omega'(I)|} \delta(I - I_c), \]

where \( H(x) \) is the Heaviside function and \( I_c \) is the single positive root of

\[ k\omega(I_c) = |x|. \]

(32)

\( P \) indicates that the principal value is to be taken and \( \delta \) is the Dirac delta function. The above formula relies on the fact that \( k > 0 \) and \( \omega(I) > 0 \). When (32) has no real positive solution, the principal value is not needed and the delta function makes no contribution.

Our Plemelj formula shows that

\[ \lim_{\gamma \to 0^+} T_k(x \pm iy) = T_{k,1}(x) \pm iT_{k,2}(x), \]

where

\[ T_{k,1}(x) = \int_{0}^{\infty} \left( \frac{f_{-k}(I)g_k(I,0)}{k\omega(I) - x} - \frac{f_k(I)g_{-k}(I,0)}{k\omega(I) + x} \right) \, dI, \]

\[ T_{k,2}(x) = \begin{cases} \frac{\pi}{k|\omega'(I_c)|} (f_{-k}(I_c)g_k(I_c,0)H(x) + f_k(I_c)g_{-k}(I_c,0)H(-x)), & I_c > 0 \text{ exists}, \\ 0, & \text{otherwise}, \end{cases} \]
and \( f \) denotes the principal value integral. Therefore we have

\[
\lim_{y \to 0^+} T(x \pm iy) = T_1(x) \pm iT_2(x),
\]

where

\[
T_1(x) = \sum_{k=1}^{\infty} T_{k,1}(x) \quad \text{and} \quad T_2(x) = \sum_{k=1}^{\infty} T_{k,2}(x).
\]

We also find

\[
\lim_{y \to 0^+} D_k(x \pm iy) = D_{k,R}(x) \pm iD_{k,I}(x),
\]

where

\[
D_{k,R}(x) = \int_0^{\infty} \frac{2k^2 f_k(I) f_{-k}(I)\omega(I)\rho_0'(I)\,dI}{k^2\omega^2(I) - x^2}
\]

and

\[
D_{k,I}(x) = \begin{cases} \frac{\pi \text{sgn}(x) f_k(I_c) f_{-k}(I_c)\rho_0'(I_c)}{|\omega'(I_c)|}, & I_c > 0 \text{ exists}, \\ 0, & \text{otherwise}. \end{cases}
\]

Thus,

\[
\lim_{y \to 0^+} D(x \pm iy) = D_R(x) \pm iD_I(x),
\]

where

\[
D_R(x) = 1 - \varepsilon \sum_{k=1}^{\infty} D_{k,R}(x) \quad \text{and} \quad D_I(x) = \varepsilon \sum_{k=1}^{\infty} D_{k,I}(x).
\]

Using these results, we find that when \( D(z) \) has no zeros

\[
j(t) = \int_{-\infty}^{\infty} U(x)e^{-ixt}\,dx,
\]

where

\[
U(x) = \frac{D_RT_2 - T_1D_I}{\pi [D_R^2 + D_I^2]}.
\]

**Claim.** If the previously stated assumptions on the initial conditions are satisfied, and \( D(z) \) has no zeros, then \( j(t) \to 0 \) as \( t \to \infty \).

**Proof.** We prove this claim using our assumptions combined, with standard estimates to show that \( D_R(x) \) and \( T_1(x) \) are bounded. Since \( D(z) \) has no zeros, we have

\[
|U(x)| \leq C \left( |T_2(x)| + |D_I(x)| \right).
\]
It then follows, using our assumptions, that

\[
\int_{-\infty}^{\infty} |U(x)| \, dx < \infty.
\]  

(34)

Therefore, by the Riemann–Lebesgue lemma, \( j(t) \to 0 \) as \( t \to \infty \). \( \Box \)

3.1.2. Asymptotic behavior

If \( D(z) \) has any zeros, then it must have a zero with \( \text{Im}(z) \geq 0 \). Suppose \( D(z) \) has zeros with \( \text{Im}(z) > 0 \). Let \( \bar{z}_M \) be a zero with the largest imaginary part, then \( -\bar{z}_M \) will also be a zero having the same imaginary part. The asymptotic behavior of \( j(t) \) is then given by

\[
j(t) \sim -\frac{T(\bar{z}_M)}{D'(\bar{z}_M)} e^{-i\bar{z}_M t} - \frac{T(-\bar{z}_M)}{D'(-\bar{z}_M)} e^{i\bar{z}_M t} \quad \text{as } t \to \infty.
\]

Thus \( j(t) \) will grow exponentially fast.

We have already shown that \( j(t) \to 0 \) as \( t \to \infty \) when \( D(z) \) has no zeros. We would like to understand the exact nature of the relaxation of \( j(t) \) to zero. We first note that when \( D(z) \) has no zeros, \( j(t) \) is the Fourier transform of \( U(x) \) and its behavior, as \( t \to \infty \) is determined by the singular behavior of \( U(x) \). First consider \( \rho_0(I) \) and \( g(I, \theta) \) to be \( C^\infty \) functions. It then follows that \( U(x) \) is also \( C^\infty \) everywhere except at \( x = \pm \omega_0 \). We also assume \( g_k(0, 0) \neq 0 \).

An asymptotic analysis shows

\[
D_{k,1}(\pm k\omega_0 + u) \sim C u^m H(\pm \text{sgn}(\omega') u) \quad \text{as } u \to 0,
\]

\[
D_{k,R}(\pm k\omega_0 + u) \sim \sum_{j=0}^{m-1} b_j u^j + C u^m \log |u| \quad \text{as } u \to 0,
\]

\[
T_{k,2}(\pm k\omega_0 + u) \sim C u^{k/2} H(\pm \text{sgn}(\omega') u) \quad \text{as } u \to 0,
\]

and

\[
T_{k,1}(\pm k\omega_0 + u) \sim \sum_{j=0}^{k-1} b_j u^j + C u^{k/2} H(\mp \text{sgn}(\omega') u) \quad \text{as } u \to 0,
\]

where \( m = k \) if \( \rho_0'(0) \neq 0 \) and \( m = k + 1 \) if \( \rho_0'(0) = 0 \). \( C \) and \( b_j \) are constants that do not depend on \( u \) and are different in each expression. We have used (27) to derive these results.

In collecting these results we see that the dominant singularities occur at \( x = \pm \omega_0 \). The behavior of \( U(x) \) near \( x = \pm \omega_0 \) is

\[
U(\pm \omega_0 + u) \sim U(\pm \omega_0) + \sqrt{|u|} (AH(u) + BH(-u)) \quad \text{as } u \to 0,
\]

(35)

where \( A \) and \( B \) are constants independent of \( u \).

Remark. The square root singularity ultimately comes from the square root dependence in the action–angle transformation which leads to (27).

The long time behavior can then be deduced from formulas found in Lighthill’s book on Fourier transforms [22]. Consequently, its long time behavior is given by

\[
j(t) \sim \frac{C}{t^{3/2}} \cos(\omega_0 t + \beta) \quad \text{as } t \to \infty,
\]
where $C$ and $\beta$ are constants independent of $t$. $C$ will be nonzero if $g_0(0, 0) \neq 0$. It is possible to compute a formula for $C$ but the formula is rather complex. In Section 4 we present a simple example where a formula for $C$ and $\beta$ is presented. If we relax the conditions, $\rho_0(I)$ and $g(I, \theta)$ are $C^\infty$ to $C^2$, it is possible to show that the above decay rate will be the same.

**Remark.** Conditions (19) and (23), with $N = 1$, ensure that $D_R$ and $T_1$ are bounded at $x = 0$. These conditions with $N = 2$ guarantee that the first derivatives will be bounded at $x = 0$. Hence the above decay rate will be the same when (19) and (23) are satisfied only for $N = 1$ and 2.

Therefore, for a wide class of initial conditions, $j(t)$ will decay like $t^{-3/2}$ even if the initial conditions can be analytically continued as entire functions. This is in contrast to the Vlasov–Poisson equation and Kuramoto’s model, where the long-time behavior depends on the initial conditions. For example, in these problems, if the initial data are entire, one observes exponential decay as $t \to \infty$. As pointed out in [3,15,16], both the Vlasov–Poisson equation and Kuramoto’s model have exponential decay on an intermediate time scale. This exponential decay is sometimes called Landau damping after Landau who first discovered it [11]. It is also connected with the asymptotic stability of solitary waves (see [26–28]).

It turns out that $j(t)$ can also decay exponentially fast on an intermediate time scale. This happens when $D(z)$ becomes very small on the real axis. This results in $U(x)$ becoming large, giving rise to an exponential decay in $j(t)$. We shall prove this below and explain precisely what is meant by an intermediate time scale. We start with the following claim:

**Claim.** Let

$$r(t) = \int_{-\infty}^{\infty} \frac{h(x)}{f(x)} e^{-ixt} \, dx,$$

where $h(x)/f(x)$ is an $L^1(\mathbb{R})$ function with $h(x)/f(x) \to 0$ as $|x| \to \infty$. Furthermore we take $f$ and $h$ to be $C^3$ functions at $x = 0$. $f$ has a single minimum at $x = 0$, with $f(0) = \mu^2$ and $h(0) = \mu$; then

$$r(t) \sim \frac{\pi}{a^2} (a - is)e^{-\mu t/a} \quad \text{for } t = O(\mu^{-1}) \quad \text{as } \mu \to 0,$$

where $a^2 = f''(0)/2$ and $s = h'(0)$.

**Proof.** Let $t = T/\mu$ and

$$\delta_\mu(x) = \frac{\mu + ix}{\mu^2 + a^2x^2},$$

then

$$r(t) = \int_{-\infty}^{\infty} \delta_\mu(x)e^{-ixT/\mu} \, dx + \int_{-\infty}^{\infty} l(x, \mu)e^{-ixT/\mu} \, dx,$$

where

$$l(x, \mu) = \frac{h(x)}{f(x)} - \delta_\mu(x).$$
One can prove that

\[
\lim_{\mu \to 0} l(x, \mu) = l_0(x),
\]

where \(l_0(x)\) is a function that is locally absolutely integrable. The key point here is the cancellation of the delta function singularities of \(\delta_\mu\) and \(h/f\) that occur as \(\mu \to 0\). Furthermore it is true that

\[
l_0(x) = O(|x|^{-1}) \quad \text{as } x \to \infty.
\]

It then follows, from a generalization of the Riemann–Lebesgue lemma (see [22]), that

\[
\lim_{\mu \to 0} \int_{-\infty}^{\infty} l(x, \mu) e^{-ixT/\mu} \, dx = 0.
\]

The first integral in \(r(t)\) can be calculated exactly, and we find

\[
\lim_{\mu \to 0} r(t) = \frac{\pi}{a^2} (a - is) e^{-T/a}.
\]

We replace \(T\) by \(\mu t\) and the claim follows. \(\square\)

We shall now use this result to prove that \(j(t)\) can decay to zero exponentially fast on an intermediate time scale. To show this result applies to the present case, we let \(\tau(x) = D_\mu \mathcal{E}_2 - T_1 D_I\) and \(d(x) = D_\mu^2(x) + D_I^2(x)\). Suppose that \(d(x)\) has a single local minimum for \(x > 0\) at \(x_0\). Since \(d(x)\) is even, it must have a local minimum at \(-x_0\). Let \(d(x_0) = \mu^2\); it thus follows that \(\tau(\pm x_0) = O(\mu)\). It is now clear that the hypotheses of the above claim are satisfied for (33). Therefore

\[
j(t) \sim e^{-\mu t/a} \left[ A(x_0) e^{-i\omega t} + A(-x_0) e^{i\omega t} \right], \quad t = O(\mu^{-1}) \quad \text{as } \mu \to 0,
\]

where

\[
A(x_0) = \frac{\tau(x_0)}{a \mu} - i \frac{\tau'(x_0)}{a^2} \quad \text{and} \quad a^2 = \frac{1}{2} d''(x_0).
\]

Remark. The behavior displayed by (36) is as if \(D(z)\) had zeros at \(\pm x_0 - i\mu/a\). We know that since \(\lim_{t \to \infty} j(t) \to 0\), that \(D(z)\) cannot have any zeros. In Landau’s analysis these would be zeros of the analytical continuation of \(D(z)\) across the real axis.

3.2. Microscopic behavior

In this section we examine the behavior of \(g(\theta, I, t)\). The main result here shows us that when \(j(t) \to 0\) the system will relax back (weakly) to the desynchronized state. This will be proved using ideas previously discussed in Section 3.1. We begin with

\[
g_k(I, t) = \frac{1}{2\pi i} \int_{-\infty+i\gamma}^{\infty+i\gamma} G_k(z) e^{-izt} \, dz, \quad (37)
\]

where

\[
G_k(z) = \frac{1}{z - \epsilon k\omega(I)} \left( g_k(I, 0) + \epsilon k f_k(I) \rho_0(I) \frac{T(z)}{D(z)} \right)
\]
this is obtained by taking the inverse Laplace transform of $G_k(I,s)$. Next we take $\gamma \to 0$, while deforming the contours around the poles, to find

$$g_k(I,t) = \sum_{j=1}^{M^+} \frac{k f_k(I) \rho_0'(I)}{z_j - k \omega(I)} \frac{T(z_j)}{D'(z_j)} e^{-iz_jt}$$

$$- \int_{-\infty}^{\infty} [g_k(I,0) + \varepsilon k f_k(I) \rho'(I) U^+(x)] \frac{1}{x - k \omega(I)} e^{-ixt} dx$$

$$+ \frac{1}{2} [g_k(I,0) + \varepsilon k f_k(I) \rho_0'(I) U^+(k \omega(I))] e^{-i\omega(I)t},$$

(38)

where

$$U^+(x) = \lim_{\gamma \to 0^+} \frac{T(x + iy)}{D(x + iy)}$$

and $z_j, j = 1, 2, \ldots, M^+$ are the zeros of $D(z)$ in the upper-half plane. We next use the identity,

$$\int \frac{e^{-ixt}}{x - \alpha} dx = -i \pi \text{sgn}(t) e^{-ixt},$$

to show that for $t > 0$,

$$g_k(I,t) = \sum_{j=1}^{M^+} \frac{k f_k(I) \rho_0'(I)}{z_j - k \omega(I)} \frac{T(z_j)}{D'(z_j)} e^{-iz_jt}$$

$$- \varepsilon k f_k(I) \rho'(I) \int_{-\infty}^{\infty} \frac{U^+(x) - U^+(k \omega(I))}{x - k \omega(I)} e^{-ixt} dx$$

$$+ [g_k(I,0) + \varepsilon k f_k(I) \rho_0'(I) U^+(k \omega(I))] e^{-i\omega(I)t}.$$  

(39)

Finally, we use

$$\int_{-\infty}^{\infty} G_k(z) e^{-izt} dz = 0 \quad \text{for} \ t > 0$$

to rewrite (39) as

$$g_k(I,t) = \sum_{j=1}^{M} \frac{k f_k(I) \rho_0'(I)}{z_j - k \omega(I)} \frac{T(z_j)}{D'(z_j)} e^{-iz_jt} - \varepsilon k f_k(I) \rho'(I) \int_{-\infty}^{\infty} W(x) e^{-ixt} dx$$

$$+ [g_k(I,0) + \varepsilon k f_k(I) \rho_0'(I) U^+(k \omega(I))] e^{-i\omega(I)t}.$$  

(40)

where

$$W(x) = \frac{U(x) - U(k \omega(I))}{x - k \omega(I)}.$$
One can use the results from Sections 4.1.1 and 4.1.2 to show that \( W(x) \) is locally absolutely integrable. Furthermore \( W(x) = O(|x|^{-1}) \) as \( |x| \to \infty \); thus we may use a form of the Riemann–Lebesgue lemma to show

\[
\lim_{t \to \infty} \int_{-\infty}^{\infty} W(x)e^{-ixt} \, dx = 0.
\]

It then follows that if \( D(z) \) has no zeros,

\[
\lim_{t \to \infty} g_k(I, t) \sim \left[ g_k(I, 0) + \varepsilon k f_k(I) \rho_0'(I)U^+(k \omega(I)) \right] e^{-ik\omega(I)t}.
\] (41)

Eq. (41) implies that as \( t \to \infty \), \( g_k \) will oscillate very rapidly in \( I \); consequently,

\[
\lim_{t \to \infty} g(I, \theta, t) = 0 \quad \text{(weakly)}.
\] (42)

Therefore, as \( j(t) \to 0 \) the system relaxes back (weakly) to the desynchronized state. Eq. (42) means the following:

\[
\lim_{t \to \infty} \int_0^\infty S(I)g(I, \theta, t) \, dI = 0,
\]

where \( S(I) \) is any \( C^\infty \) function that decays faster than \( I^{-n} \) for any \( n = 1, 2, 3, \ldots \)

4. Example

Here we present an example chosen for its computational simplicity to elucidate the ideas discussed in the previous sections. Consider the following Hamiltonian:

\[
H = \sum_{i=1}^{N} \frac{1}{2}(p_i^2 + q_i^2) + \frac{\varepsilon}{4\pi N} \sum_{i=1}^{N} \sum_{j=1}^{N} q_i q_j,
\] (43)

where \( h \) is the single-particle Hamiltonian and \( \varepsilon \) is the coupling parameter. It is convenient for us to write the Hamiltonian in terms of the action–angle coordinates of the single-particle Hamiltonian;

\[
q = \sqrt{2I \cos \theta} \quad \text{and} \quad p = \sqrt{2I \sin \theta}.
\]

In these variables the Hamiltonian takes the form given by (13) with

\[
f(\theta, I) = \sqrt{2I \cos \theta}.
\]

Therefore \( f_1(I) = f_{-1}(I) = \sqrt{I/2} \), which implies

\[
T(z) = \frac{1}{\sqrt{2}} \int_0^\infty \left( \frac{g_1(I, 0)}{\omega(I) - z} - \frac{g_{-1}(I, 0)}{\omega(I) + z} \right) \sqrt{I} \, dI
\]

and

\[
D(z) = 1 - \varepsilon \int_0^\infty \frac{I \omega(I) \rho_0'(I)}{\omega^2(I) - z^2} \, dI.
\]
Furthermore, (29) becomes
\[ j(t) = \frac{1}{\sqrt{2}} \int_0^\infty \left( g_1(I, 0)e^{-i\omega t} + g_{-1}(I, 0)e^{i\omega t} \right) \sqrt{I} \, dl \\
+ \varepsilon \int_0^t \int_0^\infty I \rho_0'(I) \sin[\omega(I)(t - \tau)]j(\tau) \, dl \, d\tau. \]  
(44)

We can use the ideas from Section 3.1.2 to deduce the following long time asymptotic expression for \( j(t) \) (provided that \( D(z) \) has no zeros):
\[ j(t) \sim \frac{\sqrt{\pi}}{D_R(\omega_0)(2|\omega'(0)|t)^{3/2}} (g_1(0, 0)e^{i\theta} + g_{-1}(0, 0)e^{-i\theta}) \quad \text{as } t \to \infty, \]  
(45)

where
\[ \theta = \omega_0 t - \frac{3}{4} \pi \text{sgn}(\omega'). \]

4.1. Stability results

Here we will find conditions that determine whether or not the desynchronized solution, \( \rho_0(I) \), is stable.

Claim. Let \( \rho_0(I) \) have a single maximum at \( I_0 \); \( D(z) \) will then have:
1. 2 zeros if \( D_R(0) < 0, \)
2. 4 zeros if \( D_R(0) > 0 \) and \( D_R(\omega(I_0)) < 0, \)
otherwise \( D(z) \) will have no zeros.

It follows then from this claim, that if \( |\varepsilon| \) is sufficiently large, \( D(z) \) will have at least one zero with \( \text{Im}(z) > 0. \) This tells us that if either of the above conditions are satisfied, the desynchronized solution will be linearly unstable. If \( D(z) \) has no zeros, the desynchronized solution is stable and \( j(t) \to 0 \) as \( t \to \infty \) for small perturbations.

If Condition (1) is satisfied, \( D(z) \) has only one zero with \( \text{Im}(z) > 0 \) and this must be purely imaginary (see Section 4.2 for the remark concerning the zeros of \( D(z) \)). This indicates that \( j(t) \) will grow exponentially in time. On the other hand, if Condition (2) is satisfied, \( D(z) \) has two zeros with \( \text{Im}(z) > 0 \), in which case we have an oscillatory instability.

Proof. Our results are obtained by recognizing \( D(z) \) to be an upper-analytic function (\( D \) is analytic for \( \text{Im}(z) > 0 \)). We then search for zeros of \( D(z) \) in the upper-half plane using the argument principle. If we find a zero of \( D(z) \), with \( \text{Im}(z) > 0 \), there will also be a zero with \( \text{Im}(z) < 0 \).

Since it is true that
\[ D(z) \sim 1 \quad \text{as } |z| \to \infty, \]
it follows that the total number of zeros of \( D(z) \) in the upper-half plane is given by the number of times the image of the real axis wraps around the origin. Let \( \mathcal{C} \) henceforth denote the real axis of the complex \( z \)-plane; the winding number of the image of \( \mathcal{C} \) under \( D \) is then
\[ M = \lim_{y \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{D'(x + iy)}{D(x + iy)} \, dx. \]
Fig. 2. The numerical solution of \( j(t) \) along with the long time asymptotic expression. The numerical solution is given by the dots and the asymptotic solution is shown by the solid line.

The image of the imaginary axis under \( D \) is given by
\[
C_D = D(C) = \{ D_R(x) + iD_I(x) \mid -\infty < x < \infty \}.
\]

In other words, \( C_D \) is a closed curve in the plane \( (D_R, D_I) \). Consider the case \( \omega'(I) < 0 \); it then follows that \( D_I(x) = 0 \) for \( |x| > \omega_0 \), \( x = 0 \), and \( x = \pm \omega(I_0) \). This information allows us to conclude that the curve \( C_D \), crosses the line \( D_I = 0 \) at \( D_R = 1 \), \( D_R(0) \), and \( D_R(\pm \omega(I_0)) \). The claim then follows using an argument similar to those found in [12,23] or [29]. A similar approach shows that the claim is also true when \( \omega'(I) > 0 \). \( \square \)

Fig. 3. The numerical solution of \( j(t) \) along with the intermediate time asymptotic expression. The numerical solution is given by the dots and the asymptotic solution is shown by the solid line.
4.1.1. A specific case

Let $\omega(I) = e^{-I}$, $\rho_0(I) = A^{-1} \exp(-\alpha(I - I_0)^2)$, and $g(\theta, I) = \rho_0(I) \cos \theta$, where $A = 2\pi \int_0^\infty \exp(-\alpha(I - I_0)^2)$. We compute a numerical solution of (44) with $\alpha = 5$, $I_0 = 0.1$, and $\varepsilon = 1.5$. The numerical solution is plotted in Figs. 2 and 3. In Fig. 2, the asymptotic solution given by (45) is plotted along with the numerical results. $d(x)$ can also be computed numerically and we find that it has a minimum at $x_0 \approx 0.810$. We also find that $\mu \approx 0.465$. Furthermore, we numerically compute the derivatives of $d$ and $\tau$ to find $a \approx 5.33$, $A(x_0) \approx 0.0412 + 0.00433i$ and $A(-x_0) \approx 0.0412 - 0.00433i$. The numerical solution and asymptotic solution given by (36) are plotted in Fig. 3.

We also present numerical simulations for the Hamiltonian given by (43) for $N = 9000$. The initial conditions are given by

$$q_k(0) = \sqrt{2I_k} \cos(\theta_k) \quad \text{and} \quad p_k(0) = \sqrt{2I_k} \sin(\theta_k),$$

where $I_k$ and $\theta_k$ are found by random sampling of the probability distributions $\rho_0(I)$ and $(1 + \eta \cos(\theta))/2\pi$. The density of the points will then be well-approximated by the solution of the kinetic equation (11) when $N \gg 1$. If $\eta = 0$ we have the desynchronized solution; when $\eta \ll 1$ the density of points will be well-approximated by the linearized kinetic equation. In these numerical simulations $a = 5$, $I_0 = 0.5$ and $\eta = 0.9$.

Fig. 4. Discrete solution: (a) $t = 0$, (b) $t = 8$, (c) $t = 16$, (d) $t = 24$, (e) $t = 32$ and (f) $t = 40$. 
In the first situation we choose $\varepsilon = 0.5$ which corresponds to a linearly stable case. The results are presented in Fig. 4, in which we have only plotted 3000 of the 9000 points used. This figure shows the formation of a spiral structure; as time increases the arms get thinner and wind around more often. This spiral structure is related to the rapid oscillations revealed by (41). As this happens the points become more uniformly distributed in phase. This means the system is relaxing to the desynchronized state.

In the second situation $\varepsilon = 4.0$. This corresponds to a linearly unstable case in which $D(z)$ has two zeros with $\text{Im}(z) > 0$; which indicates that an oscillatory instability occurs. The numerical results are shown in Fig. (5). Here we see that the points do not desynchronize but rather move in a swarm circling the origin.

Acknowledgements

The author thanks Brenda Brown for reading over the manuscript and Eduard Harabetian for a number of helpful discussions. I am extremely grateful to Harold Weitzner for pointing out references [14–16]. This work was supported in part by the National Science Foundation through a Mathematical Sciences Career grant (grant no. DMS-9625190).
References


