

The Numerical Approximation of a Delta Function

Donna Calhoun

Department of Applied Mathematics, University of Washington
Seattle, WA 98195

and

Peter Smereka

Department of Mathematics, University of Michigan
Ann Arbor, MI 48109

1 Introduction

Discrete delta functions are commonly used in many level set and front tracking applications [1, 2, 3, 4]. In particular, discrete delta functions are often used to compute geometric quantities related to interfaces that arise in such applications. For example, the volume integral of a distribution of delta function along an interface can, in theory, provide a means of computing the arclength or surface area of the interface. Recently, however, Tornberg & Engquist[5] show that delta functions based on a distance function can fail to converge even on this simple problem.

In this note we provide a new discrete delta function which our numerical experiments show computes the arclength on a given mesh with first order accuracy. Furthermore, our proposed delta has a very simple functional form and has the advantage that its support is contained within a region whose width is at most h , the dimensions of a single mesh cell.

First, we discuss Tornberg and Engquist's result in more detail. Let Γ be a curve in the plane and let $d(\mathbf{x})$ be the signed distance function to Γ . Then the length of the curve is given by

$$L = \int_{\mathbb{R}^2} \delta(d(\mathbf{x})) d\mathbf{x}. \quad (1)$$

A discrete approximation to (1) is given by:

$$L_{h,w} = \sum_{i,j} \delta_w(d_{i,j}) h^2 \quad (2)$$

where h is the mesh size and δ_w is a discrete delta function. One commonly used choice for the discrete delta function is the 1d “hat” function, given by

$$\delta_w(x) = \begin{cases} (w - |x|)/w^2 & |x| < w \\ 0 & |x| \geq w \end{cases} \quad (3)$$

where w is the width of the delta function. Tornberg & Engquist prove that for a delta function based on the hat function, there can be $O(1)$ errors in the approximation of $L_{h,w}$ to L . In particular,

they give an example in which they show that

$$\lim_{h \rightarrow 0} \frac{L_{h,h}}{L} \approx 1.12 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{L_{h,2h}}{L} \approx 1.018.$$

In the next section we describe an approach for computing a discrete delta function that does not have this problem and is easily used in level set and front tracking applications. While we show here that it provides a particularly simple way of computing the arclength or surface area, these ideas have been used for solving elliptic, parabolic and hyperbolic problems, as well as more complicated fluid flow problems and problems with discontinuous coefficients [8, 9, 10, 11, 14, 15].

2 Discrete Green's and delta functions in one dimension

The Green's function for Laplace's equation in one space dimension on the unit interval satisfies

$$\frac{d^2 g}{dx^2} = \delta(x - \alpha) \quad \text{with} \quad g(0) = g(1) = 0 \tag{4}$$

where $\delta(x)$ is the one-dimensional delta function and $0 < \alpha < 1$. It is well known that this can be replaced by the equivalent problem

$$\frac{d^2 g}{dx^2} = 0 \quad \text{with} \quad g(0) = g(1) = 0 \tag{5}$$

subject to the jump conditions at $x = \alpha$:

$$[g'(\alpha)] = 1 \quad \text{and} \quad [g(\alpha)] = 0. \tag{6}$$

Mayo [6, 7] introduced a numerical method to solve problems like (5-6) by finding the corresponding discrete version of (4). She used this to solve Laplace's and the biharmonic equation on irregular regions embedded in a regular Cartesian mesh. She first solves a boundary integral equation to obtain a density function supported on the interface, spreads this density to nearby grid points and then uses a fast Poisson solver to obtain the desired solution. The resulting solution is second order accurate, when used with a second order solver.

Since Mayo's method and others that followed, including the Immersed Interface Method, by LeVeque and Li [9], and the boundary capturing method described by Liu et. al., [12] accurately discretize problems of the form given by (5-6), it is reasonable to expect that these approaches might lead to an accurate discretization of the delta function appearing in (4).

Let consider the problem given by (5-6) in which the interface located at $x = \alpha$. Then using the ideas described by Mayo and others, one finds that the discrete version of (4) can be written as

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} = \tilde{\delta}_i \tag{7}$$

where

$$\tilde{\delta}_i = \tilde{\delta}_i^+ + \tilde{\delta}_i^- \quad (8)$$

and

$$\tilde{\delta}_i^+ = \begin{cases} (x_{i+1} - \alpha)/h^2 & \text{if } x_i \leq \alpha < x_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

$$\tilde{\delta}_i^- = \begin{cases} (\alpha - x_{i-1})/h^2 & \text{if } x_{i-1} < \alpha < x_i \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

It is easy to verify that if α does not lie exactly on a grid point, the function $\tilde{\delta}_i$ is nonzero only at the two grid points x_i, x_{i+1} for which $x_i < \alpha < x_{i+1}$. If α lies exactly on a grid point, then $\tilde{\delta}_i$ is non-zero only at $x_i = \alpha$. Furthermore, for any $\alpha, 0 < \alpha < 1$, we have

$$\sum_i \tilde{\delta}_i h = 1. \quad (11)$$

This is a basic property one would like for a discrete delta function. Furthermore we point out that $\tilde{\delta}$ is the same as δ_h in one dimension but as we shall see they differ in two dimensions.

3 Extension to two dimensions

In this section we extend the previous results to two dimensions. If an interface point $\alpha = (\alpha_x, \alpha_y)$ is between x_i and x_{i+1} then the contribution to the delta function at (x_i, y_j) can be shown to be:

$$\frac{(x_{i+1} - \alpha_x)|n_x|}{h^2}$$

where n_x is the x -component of the normal vector of the interface. Similarly, a contribution from a point y_{j+1} , for α_y between y_j and y_{j+1} is given by

$$\frac{(y_{j+1} - \alpha_y)|n_y|}{h^2}.$$

If we represent the interface as the zero level set of $\phi(x, y)$ and assume that we have a given discretization $\phi_{i,j} \equiv \phi(x_i, y_j)$ on a grid of mesh size h , then we can make the following approximations

$$x_{i+1} - \alpha_x \approx \left| \frac{\phi_{i+1,j}}{D_x^+ \phi_{i,j}} \right| \quad \text{and} \quad n_x = \frac{\partial_x \phi}{|\nabla \phi|} \approx \frac{D_x^+ \phi_{i,j}}{\sqrt{(D_x^+ \phi_{i,j})^2 + (D_y^0 \phi_{i,j})^2}}$$

where

$$D_x^+ \phi_{i,j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{h}, \quad D_x^- \phi_{i,j} = \frac{\phi_{i,j} - \phi_{i-1,j}}{h} \quad \text{and} \quad D_y^0 \phi_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2h}.$$

The advantage of using one-sided derivatives in the approximation of the normal is that the denominator will not vanish. Collecting the above results and using the same idea for the case when the interface is between x_{i-1} and x_i we obtain the following expressions

$$\tilde{\delta}_{i,j}^{(+x)} = \begin{cases} \frac{|\phi_{i+1,j}|}{h^2 \sqrt{(D_x^+ \phi_{i,j})^2 + (D_y^0 \phi_{i,j})^2}} & \text{if } \phi_{i,j} \phi_{i+1,j} \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

$$\tilde{\delta}_{i,j}^{(-x)} = \begin{cases} \frac{|\phi_{i-1,j}|}{h^2 \sqrt{(D_x^- \phi_{i,j})^2 + (D_y^0 \phi_{i,j})^2}} & \text{if } \phi_{i,j} \phi_{i-1,j} < 0 \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

We also derive similar expressions for $d_{i,j}^{(+y)}$ and $d_{i,j}^{(-y)}$. The natural extension of our 1d-delta function to two dimensions is then given by

$$\tilde{\delta}(\phi_{i,j}) = \tilde{\delta}_{i,j}^{(+x)} + \tilde{\delta}_{i,j}^{(-x)} + \tilde{\delta}_{i,j}^{(+y)} + \tilde{\delta}_{i,j}^{(-y)}. \quad (14)$$

4 Examples

We consider four examples. The first, motivated by Tornberg & Engquist, is the square whose sides are a 45° to the x and y axis. A level set representation for this interface is

$$\phi_s(x, y) = \begin{cases} \theta(x, y) & \text{if } |x - y| > 1 \text{ and } |x + y| > 1 \\ \psi(x, y) & \text{otherwise} \end{cases}$$

where

$$\theta(x, y) = \min \left(\sqrt{(x-1)^2 + y^2}, \sqrt{(x+1)^2 + y^2}, \sqrt{x^2 + (y-1)^2}, \sqrt{x^2 + (y+1)^2} \right)$$

and

$$\psi(x, y) = (|x| + |y| - 1) / \sqrt{2}$$

In this case, $\phi_s(x, y)$ is also the signed distance function. The exact length of the interface is $4\sqrt{2}$. We computed the length of this curve using the old delta function (3) and the new delta function (14).

The results are summarized in Table 1 and Figure 1. The results show that as the mesh is refined, the computed answer does not converge using the old delta function, as discussed by Tornberg & Engquist. In fact, we recover the same value for the relative error for $w = h$. The error in this example is reduced when the delta function's support is increased as shown in the table. However, we see that using our delta function (14) one obtains first order convergence to the exact

Relative error – Square			
mesh size	new	old $w = h$	old $w = 3h$
.1	1.58×10^{-2}	1.00×10^{-1}	9.45×10^{-3}
.05	7.91×10^{-3}	1.11×10^{-1}	3.82×10^{-4}
.025	3.96×10^{-3}	1.16×10^{-1}	5.29×10^{-3}
.0125	1.98×10^{-3}	1.19×10^{-1}	7.75×10^{-3}
.00625	9.89×10^{-4}	1.20×10^{-1}	8.98×10^{-3}
.003125	4.94×10^{-4}	1.21×10^{-1}	9.59×10^{-3}
.0015625	2.47×10^{-4}	1.21×10^{-1}	9.90×10^{-3}

Table 1: Computation of the interface length for square using the new delta function (14) and the old delta function (3).

Relative error – Circle			
mesh size	new	old $w = h$	old $w = 3h$
.1	5.98×10^{-3}	1.89×10^{-2}	3.55×10^{-4}
.05	2.57×10^{-3}	5.07×10^{-3}	3.20×10^{-4}
.025	9.92×10^{-4}	6.91×10^{-3}	5.63×10^{-4}
.0125	7.46×10^{-4}	5.17×10^{-3}	3.78×10^{-4}
.00625	2.82×10^{-4}	7.55×10^{-4}	1.12×10^{-4}
.003125	7.09×10^{-5}	3.06×10^{-3}	1.91×10^{-4}
.0015625	2.82×10^{-5}	2.51×10^{-3}	2.31×10^{-4}

Table 2: Computation of the interface length for a circle using the new delta function (14) and the old delta function (3).

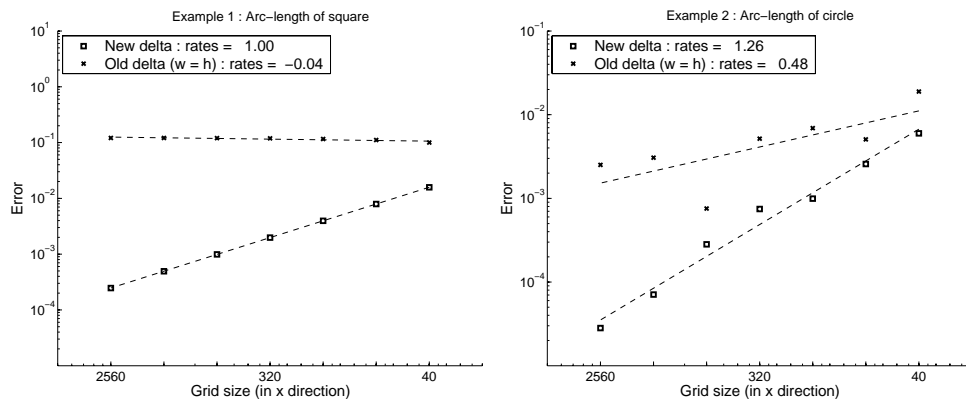


Figure 1: The relative errors for curve length are plotted as function grid size using the old delta function with $w = h$ and the new delta function for both the square and the circle

Relative error – Line Integral			
mesh size	new	old $w = h$	old $w = 3h$
.1	1.54×10^{-4}	3.62×10^{-2}	1.50×10^{-1}
.05	7.37×10^{-4}	8.84×10^{-3}	3.75×10^{-2}
.025	4.91×10^{-4}	5.78×10^{-3}	8.72×10^{-3}
.0125	6.26×10^{-4}	5.25×10^{-3}	2.72×10^{-3}
.00625	2.51×10^{-4}	7.85×10^{-4}	5.00×10^{-4}
.003125	6.28×10^{-5}	3.04×10^{-3}	4.26×10^{-5}
.0015625	2.63×10^{-5}	2.51×10^{-3}	2.62×10^{-4}

Table 3: Computation of the line integral using the new delta function (14) and the old delta function (3).

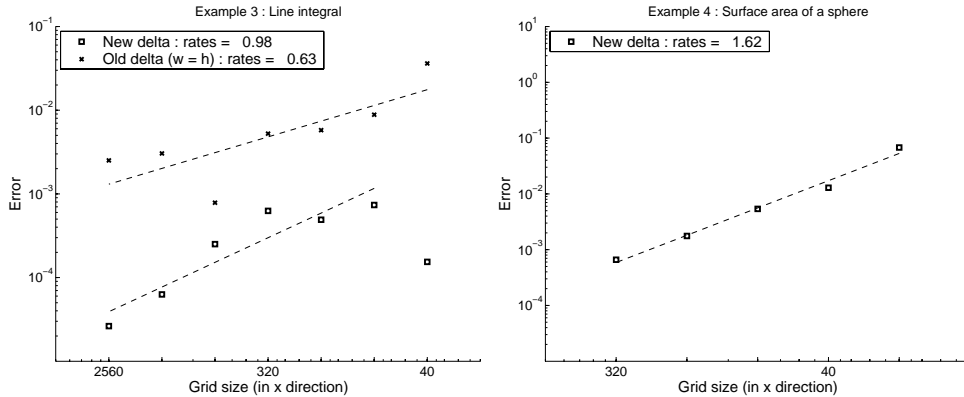


Figure 2: The relative error is plotted for the line integral as a function of grid size using the new delta function and the old delta function with $w = h$

value. We remind the reader that the support of our delta function is only one grid point on either side of the interface.

The next case we study is the circle. Here we use the level set function given by

$$\phi_c(x, y) = \sqrt{x^2 + y^2} - 1. \tag{15}$$

As in the previous case, ϕ_c is the signed distance function. The results are shown in Figure 1 and Table 2. At first sight it appears that each method provides an accurate answer. Nevertheless, a closer examination shows that answers obtained using (3) for the delta function converge at a rate much less than first order if at all. Our method provides a more accurate approximation and its rate convergence is clearly at least first order.

Next, we present example illustrating the advantage of having a narrowly support delta function. Here, we compute a line integral of a function on a circle using discrete delta functions. Namely,

$$\int_{x^2+y^2=1} f(x, y) ds = \int \delta(\phi_c(x, y)) f(x, y) dx dy$$

where ϕ_c is given by (15) and $f(x, y) = (3x^2 - y^2)(x^2 + y^2)$. It is easy to show that its value is 2π and our computations are shown in Table 3 Figure 2. The result using (3) with $w = 3h$ is much worse, for coarse meshes, than either of the others due to the variation of $f(x, y)$. Clearly, once again, the delta function given by (14) provides a good approximation with first order convergence.

Finally we consider an example in three dimensions where ϕ is not a distance function, namely

$$\phi_3 = (x^2 + y^2 + z^2 - 1)(x^2 + 10y^2 + z^4)$$

Here the zero level set of ϕ_3 is a sphere of unit radius. The results are shown on Figure 2. As the results show, the method has an experimental order of convergence that is between 1 and 2.

5 Summary

We have proposed a new discrete delta function which is very narrowly supported and shows experimental first order accuracy for computing integrals. It would be interesting to provide a proof that this method is first order accurate and to develop a second order version. Finally, we point out that Engquist, Tornberg and Tsai[13] have also developed a delta function approximation for level set calculations, but we believe that ours is simpler.

Acknowledgments

P. Smereka was supported in part by the National Science Foundation through Grants No. DMS-0207402. D. Calhoun was supported by DOE grant DE-FC02-01ER25474.

References

- [1] S.O. Unverdi and G. Trygvasson, A front-tracking method for the computation of multiphase flow, *J. Comput. Phys.***100** 25-37 (1992)
- [2] M. Sussman, P. Smereka, and S. Osher A level set method for computing solutions to incompressible two phase flow, *J. Comput. Phys.* **119**, 146-159 (1994).
- [3] S. Osher and R. Fedkiw, *Level Set Methods and Dynamic Implicit Surfaces*, Springer, New York (2000).
- [4] D. Peng, B. Merriman, S.J. Osher, H.K. Zhao, M.J. Kang A PDE-based fast local level set method, *J. Comput. Phys.* **155** 410-438 (1999).
- [5] A.K. Tornberg and B. Engquist, Numerical approximations of singular source terms in differential equations, UCLA CAM Report (2004)
- [6] A. Mayo, The fast solution of Poisson's and the biharmonic equations on irregular regions, *SIAM J. Sci. Comput.* **21** 285-299 (1984).
- [7] A. Mayo, The rapid evaluation of volume integrals of potential theory on general regions, *J. Comput. Phys.***100** 236-245 (1992).
- [8] , M. Lai, Z. Li, Immersed Interface Method for the Navier-Stokes Equations with Singular Forces, *J. of Comput. Phys*, **171**, 822-842 (2001).
- [9] R.J. LeVeque and Z.L. Li, Immersed interface methods for Stokes flow with elastic boundaries, *SIAM J. Sci. Comput.***18** 709-735 (1997).
- [10] Z. Li, H. Zhao, S. Osher, A hybrid method for moving interface problems with application to the Hele-Shaw flow, *J. of Comput. Phys*, **134**, 236-252 (1997).

- [11] D. Calhoun, A Cartesian grid method for solving the two-dimensional streamfunction-vorticity equations in irregular regions, *J. of Comput. Phys*, **176**, 231-275 (2002).
- [12] X.D. Liu, R. Fedkiw, M. Kang, A boundary condition capturing method for Poisson's equation on irregular domains, *J. Comput. Phys.***160** 151-178 (2000).
- [13] B. Engquist, A.K. Tornberg, and R. Tsai, Discretization of Dirac delta functions in level set methods, UCLA CAM Report (2004).
- [14] A. Wiegmann, K. P. Bube, The Explicit-jump Immersed Interface Method : Finite Difference Methods for PDEs with Piecewise Smooth Solutions, *SIAM J. Of Num. Anal.*, **37**, No. 3, 827-862, (2000).
- [15] A. Wiegmann, K. P. Bube, The Immersed Interface Method for Nonlinear Differential Equations with Discontinuous Coefficients and Singular Sources, *SIAM J. of Num. Anal.*, **35**, No. 1, 177-200, (1998).