Semi-Implicit Level Set Methods for Curvature and Surface Diffusion Motion

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Reference: P. Smereka, J. Scientific Computing, **19**, 439-456 (2003).

THE LEVEL SET METHOD

To fix ideas consider a surface in R^3

Γ = Interface (A two dimensional object)

$u(\mathbf{x}, t)$ = Level set function (A scalar function with $\mathbf{x} \in R^3$)

Γ is implicitly defined using the level set function:

$$\Gamma = \{\mathbf{x} | u(\mathbf{x}, t) = 0\}.$$

LEVEL SET EVOLUTION

$s_n = s_n(\alpha, \beta)$ is the interface velocity α and β are coordinates on Γ.

$v_n = v_n(\mathbf{x})$ is an extension of s_n. This has the property

$$v_n(\mathbf{x})|_{\mathbf{x}\in\Gamma} = s_n$$

The time evolution of u is then given by

$$\frac{\partial u}{\partial t} + v_n |\nabla u| = 0$$

Developed by Osher & Sethian (1989).

MEAN CURVATURE FLOW

Consider the motion of interface whose normal speed is $-\kappa$

Then $s_n = -\nabla_s \cdot \mathbf{n}$

But the normal to the interface is $\mathbf{n} = \dfrac{\nabla u}{|\nabla u|}$

One can show that $v_n = -\nabla \cdot \mathbf{n}$

The evolution equation is:

$$\frac{\partial u}{\partial t} - \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) |\nabla u| = 0$$

NUMERICAL ISSUES

- Level set equation is solved on fixed grid
- Interface is “captured” using a contour plotter
- Curvature flow is analogous to solving a nonlinear heat equation
- Curvature flow problem is stiff

MOTION BY SURFACE DIFFUSION

Here $s_n = \Delta_s \kappa$ where Δ_s is the surface Laplacian

This maybe written in terms of the level set function as follows

1. Surface divergence is $\nabla_s = \nabla - \mathbf{n}\partial_n$.
2. $\partial_n = \mathbf{n} \cdot \nabla$.
3. $\nabla_s = (I - \mathbf{nn})\nabla$.
4. $P_\mathbf{n} = I - \mathbf{nn}$ is a projection operator
5. $\Delta_s = (P_\mathbf{n}\nabla)^2$
6. Finally recall $\mathbf{n} = \nabla u / |\nabla u|$

MOTION BY SURFACE DIFFUSION

Since $\Delta_s \kappa$ in terms of u we can write the time evolution of the level set function for surface diffusion

$$\frac{\partial u}{\partial t} + \Delta_s \kappa |\nabla u| = 0$$

- Level set methods for surface diffusion have been developed by Chopp & Sethian and Khenner et al
- Numerically, Surface diffusion problems are very stiff
- In this respect it is analogous to solving $u_t = -u_{xxxx}$
- For an explicit method, stability requires that $k < Ch^4$;. $k =$ time step and $h =$ mesh size.

MORE ISSUES

- Since the evolution equation is nonlinear it is not easy to use implicit methods. Chopp & Sethian and Khenner et al use explicit methods are used and very small time steps.
- As pointed out by Chopp & Sethian, this evolution equation does not have a maximum principle, consequently an embedded curve may not remain so. In particular, new interfaces may spontaneously nucleate
- Chopp & Sethian have successfully ameliorated this problem by using narrow band level set methods and thereby only computing near the interface

SEMI-IMPLICIT METHODS FOR CURVATURE FLOW

- A simple identity $|\nabla u|\kappa = \Delta u - N(u)$
- where $N(u) = \frac{\nabla u}{|\nabla u|} \cdot \nabla(|\nabla u|) = \partial_n |\nabla u|$
- Evolution equation is $u_t = \Delta u - N(u)$
- Time discretization is: $u^{n+1} = u^n + k\Delta u^{n+1} + kN(u^n)$
- Even though $N(u)$ is a second order operator we find the method is stable

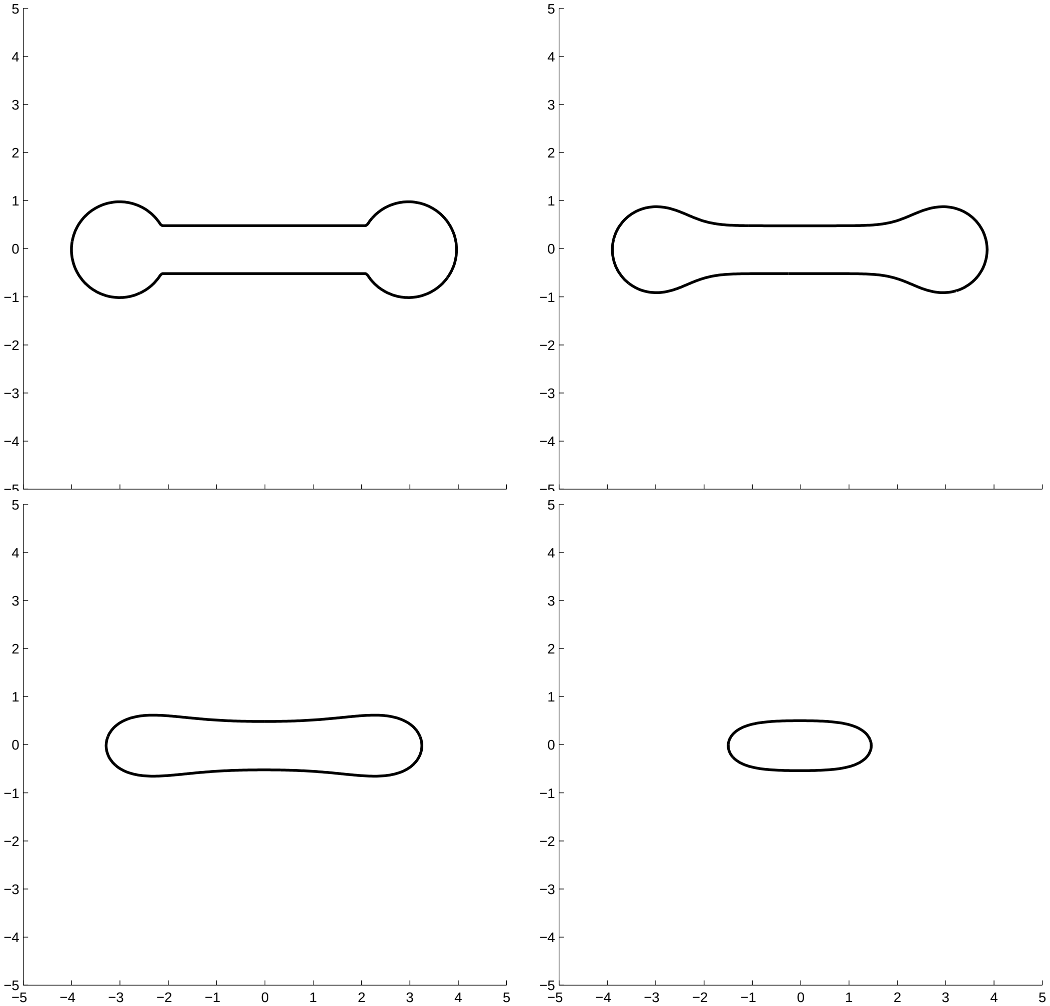

Figure 1: Motion by mean curvature

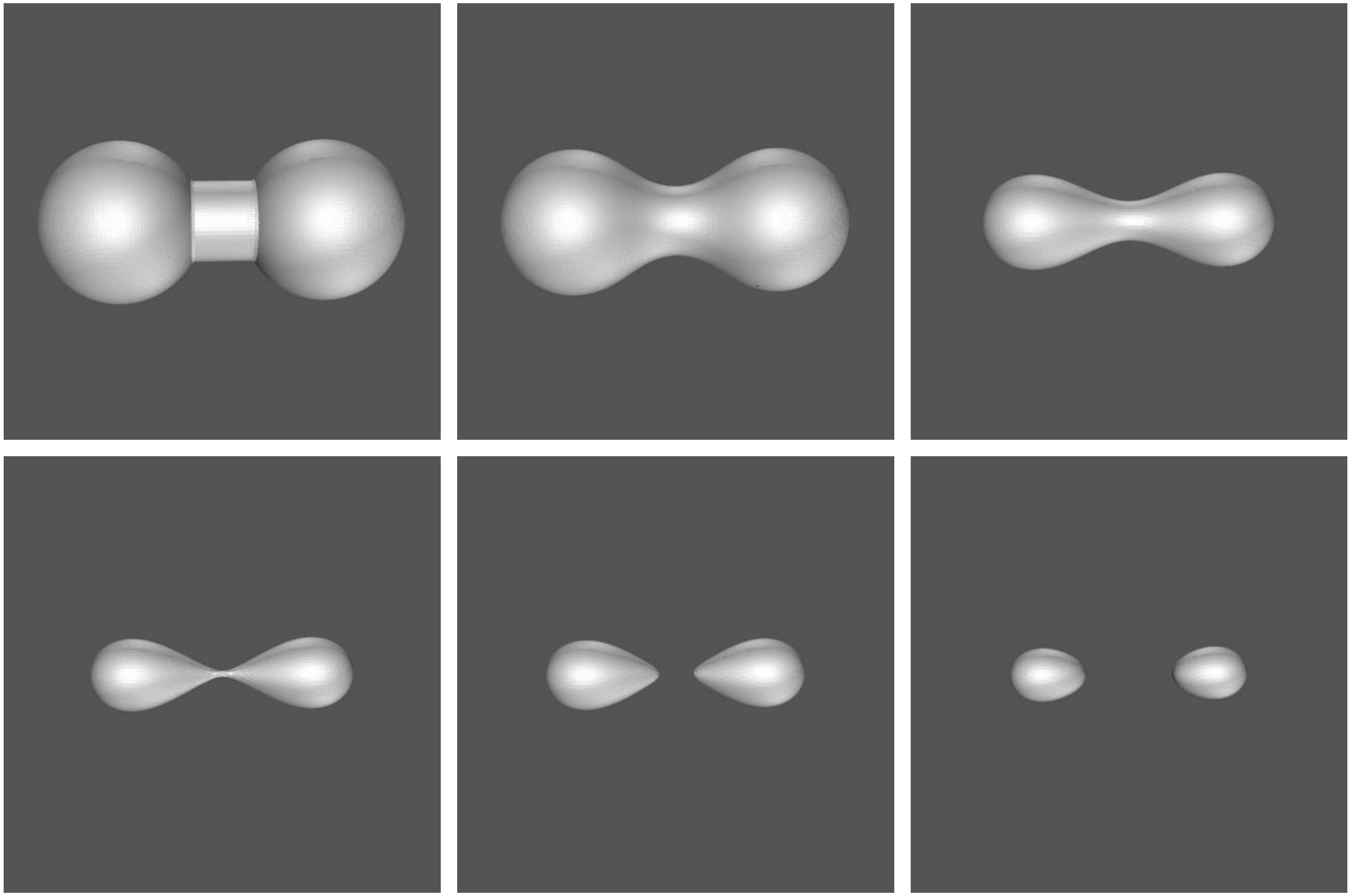

Figure 2: Motion by Mean Curvature

SEMI-IMPLICIT METHODS FOR SURFACE DIFFUSION

- Evolution Equation $u_t + S(u) = 0$ where $S(u) = \Delta_s \kappa |\nabla u|$.
- Write it as: $u_t = -\beta \Delta^2 u + \beta \Delta^2 u - S(u)$ where Δ^2 is the biLaplacian and β is a constant yet to be determined
- Dicretize as: $u^{n+1} = u^n - k\beta \Delta^2 u^{n+1} + k\left[\beta \Delta^2 u^n - S(u^n)\right]$ where k is the time step
- This equation is first order accurate in time for any value of β
- Same idea as the Laplace Modified Galerkin Method developed by Douglas & Dupont

WHY IS WORKS

- Discrete Equation can be written as:
  $u^{n+1} = u^n + (I + k\beta\Delta^2)^{-1}\,[-kS(u^n)]$
- The operator $(I + k\beta\Delta^2)^{-1}$ is a positive definite smoothing operator
- It is therefore apparent that we have essentially applied a smoothing operator to an explicit scheme
- The smoothing operator is able to suppress the unstable high wave number modes
- $(I + k\beta\Delta^2)^{-1}$ is easily and efficiently computed using a FFT

CHOICE OF β

- Consider the ode: $\dot{x} = -ax$ where $a > 0$.
- Discretizing this ode using the above idea produces the scheme $x_{n+1} = x_n - kax_n/(1 + k\beta)$
- This is stable provided $k(a - 2\beta) < 2$.
- The above is satisfied if $\beta \geq \frac{a}{2}$.
- For surface diffusion we take $\beta = \frac{1}{2}$

THE ALGORITHM

1. Initialize u

2. Set $\kappa = 0$ for all grid points.

3. Compute κ for grid points within one grid cell of the interface.

4. Extend κ away from the interface by least at 2 grid points.

5. Set $S = 0$ at all grid points.

6. Compute S for grid points within one grid cell of the interface.

7. Extend S away from the interface by at least one grid point.

8. Update u using $u^{n+1} = u^n + (I + k\beta\Delta^2)^{-1}[kS]$

9. One step has been completed, return to Step 2 and repeat.

VELOCITY EXTENSION

- Solve a hyperbolic equation whose characteristics point outward from the interface and thereby carry information near the interface to the rest of the domain
- In this setting u is the level set function and c is a variable that is known near the zero level set of u. We wish to extend c off the interface in such away it is constant in normal directions
- This is done by solving the following hyperbolic equation:

$$\frac{\partial c}{\partial t} + \mathrm{sign}(u)\frac{\nabla u}{|\nabla u|} \cdot \nabla c = 0$$

REMARKS

- The advantage of computing κ and S near the interface and then extending them is that they can be poorly behaved just slightly away from the interface especially when a topology change has just occurred or one is imminent.
- Finally we remark that the operator used in Step 8 is nonlocal which can cause difficulties which will be discussed below

RESULTS

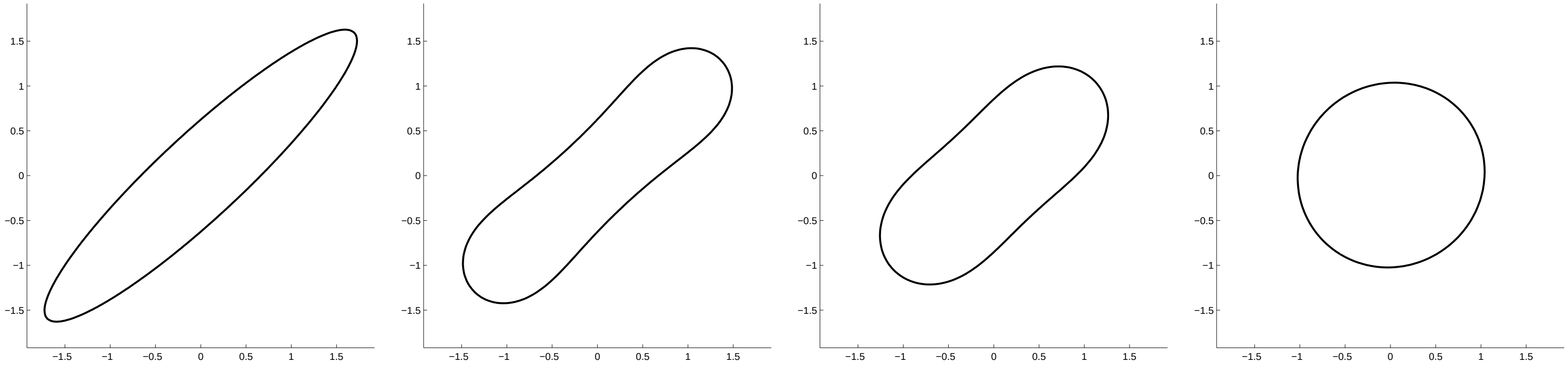

Figure 3: Surface diffusion, $h = .015$ and $k = 1 \times 10^{-4}$ $(k/h^4 \approx 2 \times 10^4)$ Chopp & Sethian's explicit scheme used $h = .06666$ and $k = 5 \times 10^{-6}$ $(k/h^4 \approx .25)$. It should also be pointed out, with this same mesh size, we could have taken time steps as large as .005 and obtained qualitatively similar results (with less accuracy).

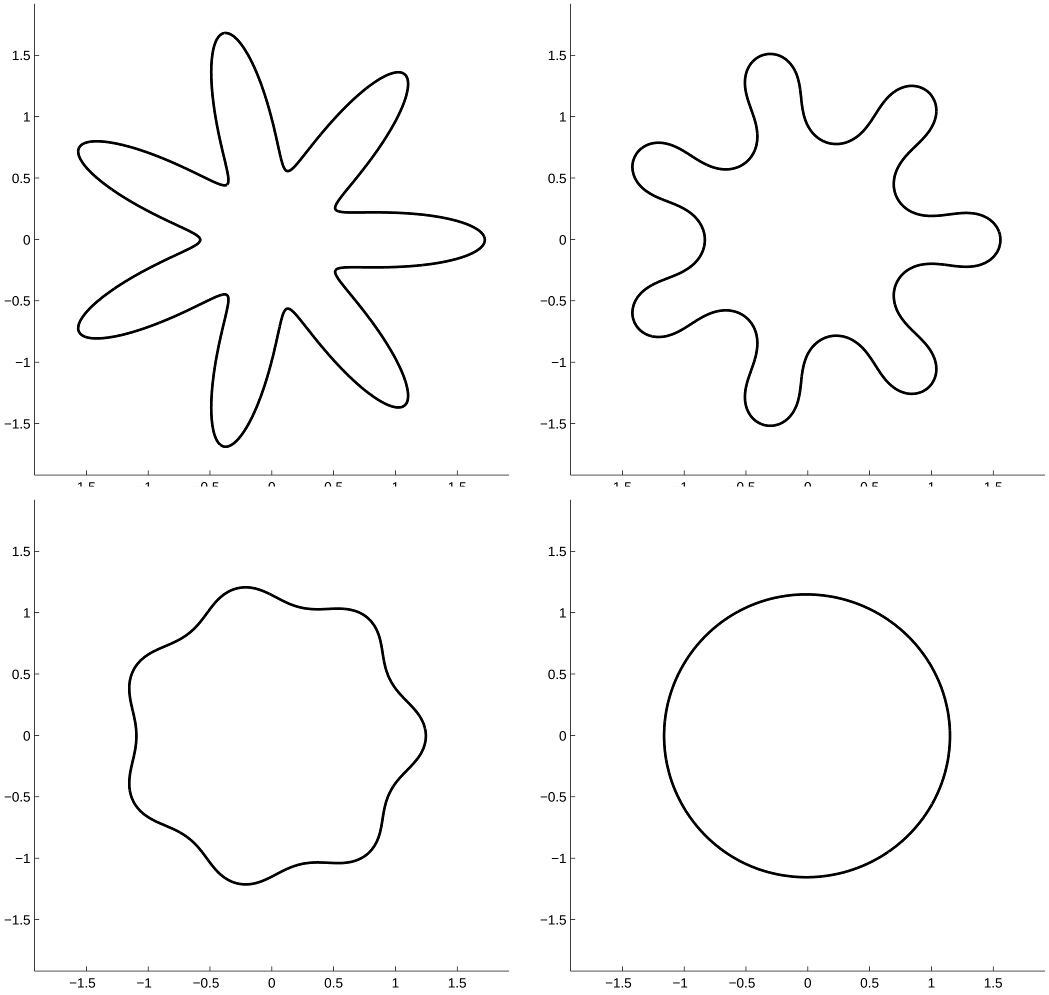

Figure 4: Surface Diffusion in Two Dimensions

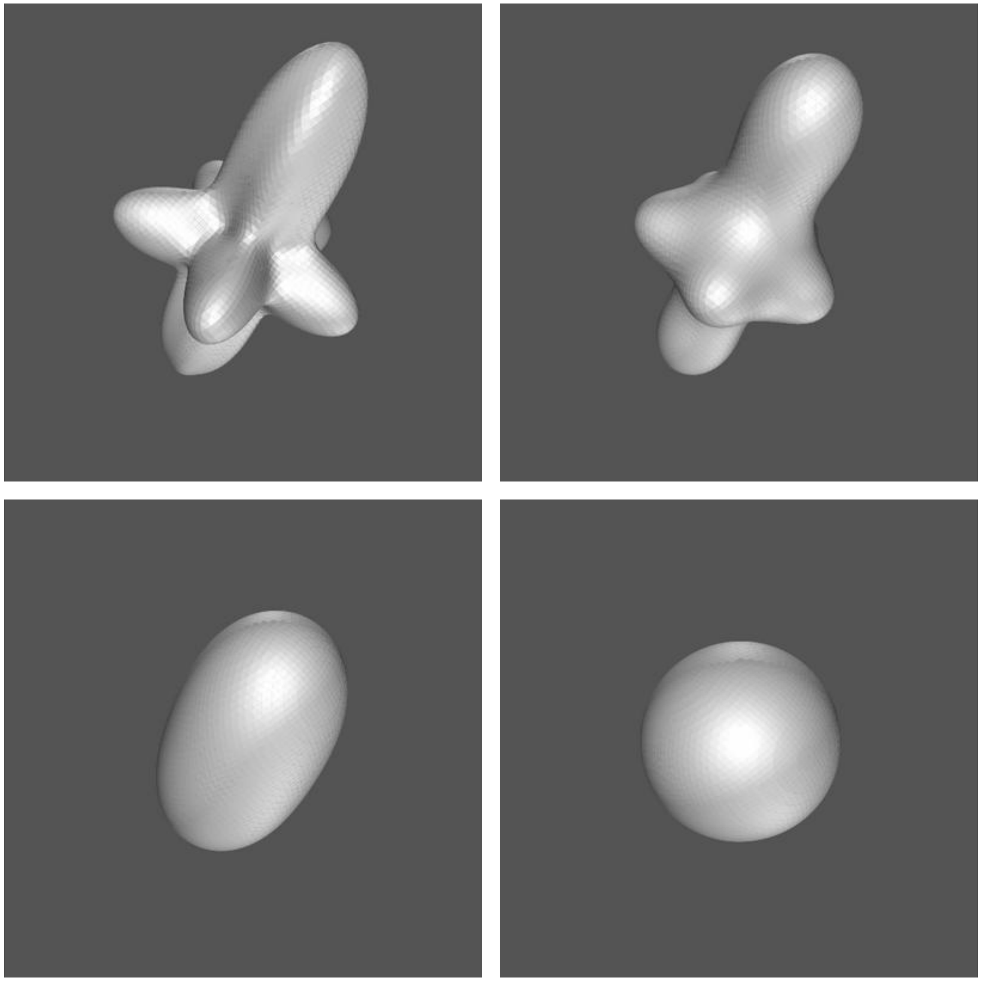

Figure 5: Surface Diffusion in Three Dimensions

PROBLEMS

1. Topology Changes
    - Merging of fronts is difficult presumably due to the stiffness of the problem and the lack of a maximum principle
    - When two fronts merge the mean curvature necessarily becomes infinite and the surface Laplacian of the mean curvature is even more singular
    - Very small time steps seem to help
2. Nonlocal Effects
    - Disconnected surfaces that should not interact do

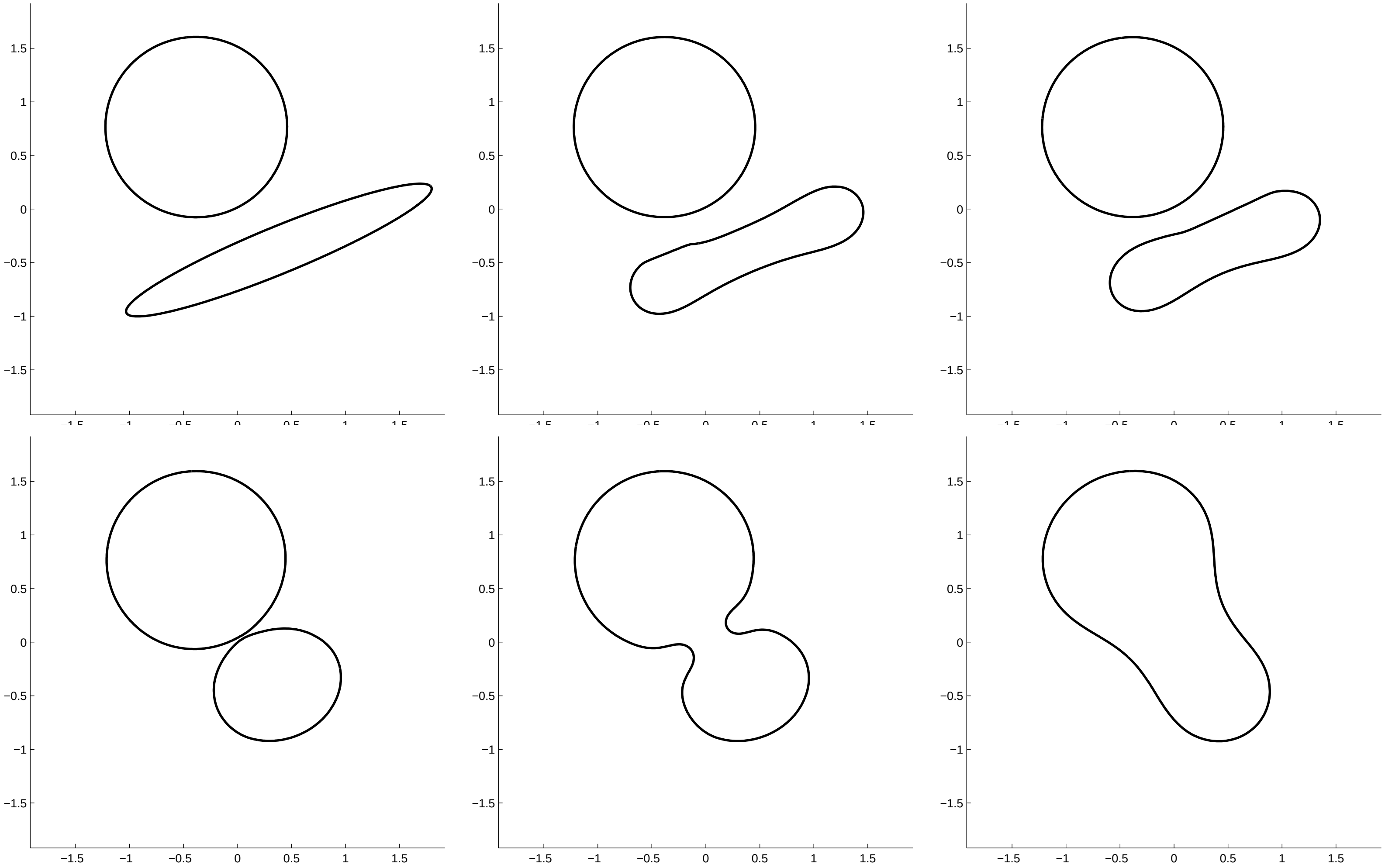

Figure 6: Surface Diffusion: Nonlocal Effects and Merging

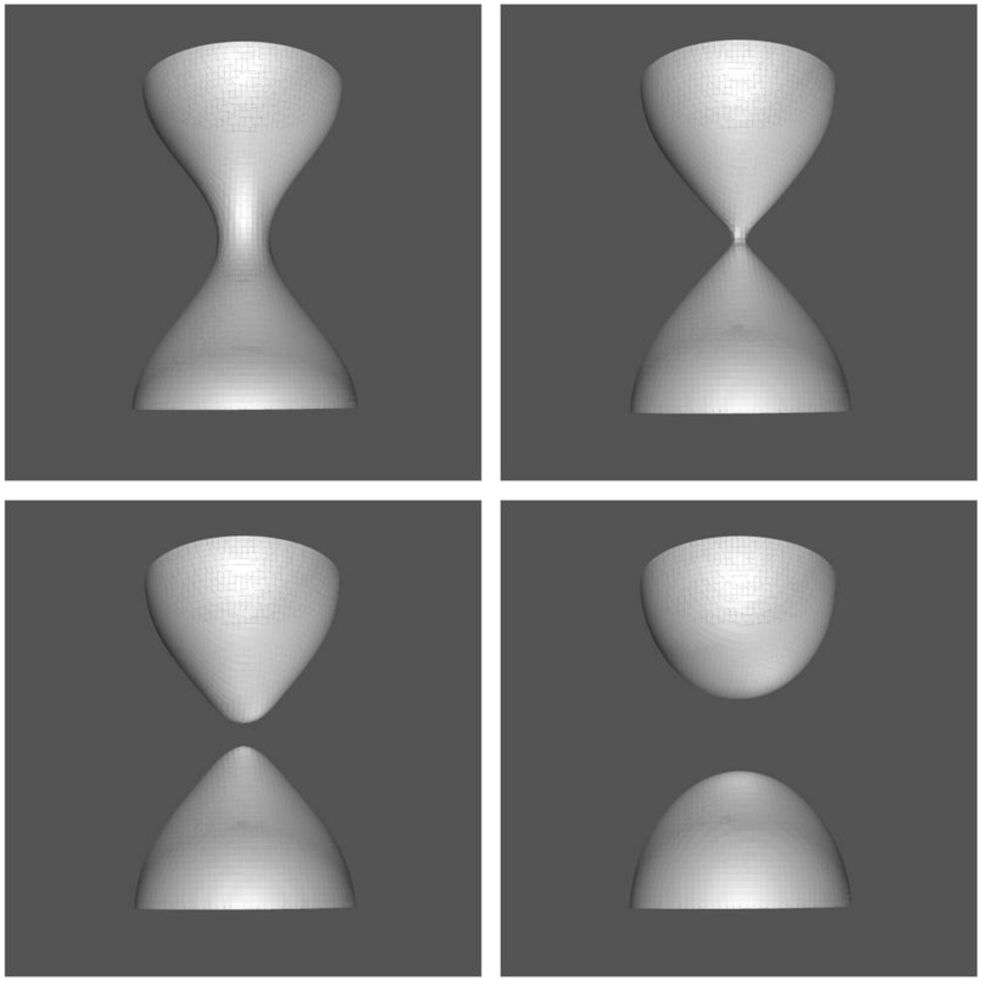

Figure 7: Surface Diffusion: Pinch Off

CONCLUSIONS

- A semi-implicit algorithm for curvature flows and surface diffusion has been developed which has the advantage of beginning no more difficult to implement than an explicit method
- The method appears to be quite stable
- The method is first order in time but can be extended to higher order using Richardson extrapolation
- The are serious issues with nonlocal effects and problems and topology changes for surface diffusion