A remark on the solitary wave stability for a Boussinesq equation

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In this paper we shall study the stability of solitary waves of a Boussinesq type equation namely

$$u_{tt} = u_{xx} + \frac{1}{1+n}(u^{n+1})_{xx} + u_{xxtt} \tag{1.1}$$

for $n = 1$ this equation has been used to study water waves, see for example, Berryman (1976) and nonlinear elastic rods by Soerensen, Christiansen & Lomdahl (1984). A similar equation also occurs in the study of pressure wave propagation in bubbly fluids, see for example Birnir & Smereka (1990, §4).

The stability of solitary waves of the KdV and BBM equations were proved by Benjamin (1972) and Bona (1974). A similar method was applied to the generalized KdV, generalized BBM and the nonlinear Schrödinger equations by Weinstein (1984, 1986, 1987) to prove orbital stability of ground state solutions. Subsequently, Grillakis, Shatah & Strauss (1987), henceforth denoted GSS, obtained sharp conditions for the orbital stability and instability of ground state solutions for a class of abstract Hamiltonian systems. Bona, Souganidis & Strauss (1987), (BSS) obtained similar results for KdV type equations, a class not considered by GSS.

In this remark we shall examine the Boussinesq equation and see that it fits into the class of abstract Hamiltonian systems studied by GSS, however it will turn out that an important hypothesis needed to apply their results cannot be satisfied. Nevertheless, numerical results tend to suggest that the condition they give for (in)stability appears to be applicable. Furthermore, Pego & Weinstein (1991) have proved the existence of at least one unstable eigenvalue of Eq. 1.1 linearized about the solitary wave when the aforementioned condition is satisfied. We shall present some reasons why we expect orbital stability in the case when the other case holds.

2. Hamiltonian Formalism and the Problem in Proving Stability. In order to deduce the Hamiltonian structure of Eq. 1.1 it is useful to consider a new variable $v_x = u$ and Eq. 1.1 can be written as

$$v_{tt} = v_{xx} + \frac{1}{1+n}(v_x^{n+1})_x + v_{xxtt} \tag{2.1}$$

The Hamiltonian for Eq. 2.1 is

$$\mathcal{H} = -\int_{-\infty}^{+\infty} \left\{ \frac{1}{2} p B^{-1} p + \frac{1}{2} v_x^2 + \frac{1}{(n+1)(n+2)} v_x^{n+2} \right\} dx \tag{2.2}$$

where $B = 1 - \partial_x^2$ and the minus sign is for sake of convenience. The equations of motion are then

$$\begin{aligned} v_t &= \frac{\delta \mathcal{H}}{\delta p} = -B^{-1} p \\ p_t &= -\frac{\delta \mathcal{H}}{\delta v} = -v_{xx} - \frac{1}{1+n} (v_x^{n+1})_x \end{aligned} \tag{2.3}$$

This is of the form studied by GSS. Another conserved quantity, associated with the translation invariance of the equations of motion, is

$$I = \int_{-\infty}^{+\infty} p v_x dx \tag{2.4}$$

This quantity is crucial in study solitary waves because they are stationary points of the functional

$$\mathcal{F} = \mathcal{H} - cI \tag{2.5}$$

It is an easy check that $\delta \mathcal{F} = 0$ if

$$\begin{aligned} B^{-1} p + c v_x &= 0 \\ v_x + \frac{v_x^{n+1}}{n+1} + cp &= 0 \end{aligned} \tag{2.6}$$

Eliminating p we find $v_x = y$ satisfies the equation

$$c^2 y'' + (1 - c^2) y + \frac{y^{n+1}}{n+1} = 0 \tag{2.7}$$

which for $c^2 > 1$ has the solution

$$y(x) = \left[\frac{1}{2} (c^2 - 1)(n+1)(n+2) \right]^{1/n} \operatorname{sech}^{2/n} \left[\frac{n(c^2-1)^{1/2} x}{2c} \right] \tag{2.8}$$

The stationary point of $\mathcal{F}$ is therefore

$$\phi = (v, p) = \left(\int_{-\infty}^{x} y(\xi) d\xi, \ -cBy \right)$$

An important quantity, to be used later, is I evaluated on the solitary wave. Using Eq. 2.4 and Eq. 2.6 this is

$$I(\phi) = -c \int_{-\infty}^{+\infty} yBydxdx = -D(n)c^2(c^2-1)^{\frac{2}{n}-\frac{1}{2}} \left[1 + \frac{n}{n+4}\frac{c^2-1}{c^2}\right]$$

where $D(n) = 2\left[\frac{1}{2}(n+1)(n+2)\right]^{2/n} \beta\left(\frac{1}{2}, \frac{2}{n}\right)$ and β is the beta function.

For wave equations studied by GSS and BSS the stability of the solitary wave was deduced by examining the second variation of $\mathcal{F}$ and showing that $\delta^2\mathcal{F} > 0$ in a neighborhood of $I = I(\phi)$. A computation of $\delta^2\mathcal{F}$ around the solitary wave gives

$$\delta^2\mathcal{F} = \int_{-\infty}^{+\infty} (\delta v_x, \delta p) \begin{pmatrix} -(1+y^p) & -c \\ -c & -B^{-1} \end{pmatrix} \begin{pmatrix} \delta v_x \\ \delta p \end{pmatrix} dx \tag{2.9}$$

This is more easily studied with the transformation

$$\begin{pmatrix} \delta v_x \\ \delta p \end{pmatrix} = \begin{pmatrix} I & 0 \\ -cB & B \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \tag{2.10}$$

which simplifies Eq. 2.9 to

$$\delta^2\mathcal{F} = \int_{-\infty}^{+\infty} (uLu - wBw)dx \tag{2.11}$$

where $Lf = -c^2\partial_x^2 f + (c^2 - 1 - y^p)f$. It is readily verified that

$$L\partial_x y = 0 \tag{2.12}$$

Since y is an even function then Eq. 2.12 implies that there is a single eigenfunction of L with a negative eigenvalue. Also, continuous spectrum of L is on the positive real axis bounded away from zero by (c^2-1), recall $c^2 > 1$. Therefore L is strictly positive except for two directions which are associated with the two degrees of freedom of the solitary wave, namely its phase and speed. Since B is a positive definite operator we see that $\delta^2\mathcal{F}$ is negative in an infinite number of directions. Two of these directions can be associated with point spectrum and the rest with a continuous spectrum.

When examining the stability of the solitary wave solution to the bubbly flow model discussed by Birnir and Smereka one finds that $\delta^2\mathcal{F}$ has the same structure as does this Boussinesq equation as shown by Birnir, Smereka, and Weinstein (1989).

In the the class of equations considered by GSS and BSS, $\delta^2\mathcal{F}$ is positive except for two directions which are again associated with the two degrees of freedom of the solitary waves. Even with this lack of definiteness it is possible to still prove stability using $\delta^2\mathcal{F}$. The basic idea is to restrict the class of perturbations to those that have the momentum, $I = I(\phi)$, and are orthogonal to the direction that gives $\delta^2\mathcal{F} = 0$. With this restriction it is possible to show

$$\delta^2\mathcal{F} > 0 \quad \text{provided} \quad \frac{dI(\phi)}{dc} < 0 \tag{2.13}$$

This crucial observation allows the proof of orbital stability to be completed. The work of BSS and GSS shows that when $dI(\phi)/dc > 0$ the solitary wave is orbitally unstable.

For the Boussinesq equation considered here, the fact that $\delta^2\mathcal{F}$ is is negative in an infinite number of directions prevents us from using the results of GSS to say anything about the stability or instability of the solitary waves. One might speculate that $\delta^2\mathcal{F}$ being so badly indefinite is indicative of some sort of instability, however numerical work suggests that when $dI(\phi)/dc < 0$ that the solitary waves are stable and when $dI(\phi)/dc > 0$ the solitary waves are unstable.

Recently, Pego & Weinstein (1991) have examined the stability of the the solitary waves for the generalized KdV, generalized BBM, and Eq. 1.1 using the Evans function. The Evans function is an function of a complex variable that is analytic in the closed right half plane and whose zeros correspond to eigenvalues of the linearized evolution equation. They shown that both gKdV and gBBM are exponentially unstable with a single distinct eigenvalue if $dI(\phi)/dc > 0$. They also that Eq. 1.1 has at least one unstable eigenvalue if $dI(\phi)/dc > 0$. (Pego and Weinstein use $\mathcal{N}$ which is $-I$) For the Boussinesq equation considered here we have

$$\frac{dI(\phi)}{dc} = -(c/n)D(n)(c^2-1)^{\frac{2}{n}-\frac{3}{2}}(c^2(4+2n)-3n) \tag{2.14}$$

which shows that

$$\frac{dI(\phi)}{dc} > 0 \quad \text{if} \quad c^2 < \frac{3n}{2n+4} \tag{2.15}$$

In a related study, Pego, Smereka, and Weinstein (1991) have numerically computed the Evans function for Eq. 1.1. The numerical evidence shows that if $dI(\phi)/dc > 0$ there is only one unstable eigenvalue and if $dI(\phi)/dc < 0$ there are no unstable eigenvalues and the only eigenvalue embedded in the continuous spectrum is the one at the origin with multiplicity two. In this problem the continuous spectrum fills the imaginary axis. It should be remarked that even if it could be proved that the spectrum of the linearized operator was entirely on the imaginary axis that would not necessarily imply the orbital stability of the solitary wave, see for example Holm, Marsden, Ratiu and Weinstein (1985).

These results suggest that the stability of the solitary wave is determined by the "bad" direction associated with the degree of freedom related to the speed of the solitary wave. This is a one dimensional space, therefore it seems that the lack of definiteness caused by the infinite other directions do not play a role in determining stability. In other words, the mechanism for stability and instability of this Boussinesq equation is exactly the same as for the systems studied by GSS and BSS even though $\delta^2\mathcal{F}$ is badly indefinite.

3. Remarks on Stability. Consider Eq. 1.1 without the nonlinearity in a frame of reference moving in the $+x$ direction with speed c, this is

$$(1-\partial_z^2)(u_{\tau\tau}-2cu_{\tau z}+c^2u_{zz})=u_{zz} \tag{3.1}$$

where (z,τ) are the moving (x,t) variables. Let us try to prove the stability of the $u=0$ solution using an energy method. Taking the Fourier transform of Eq. 3.1 gives

$$\hat{u}_{\tau\tau}-2ick\hat{u}_\tau+\left(\frac{k^2}{1+k^2}-c^2k^2\right)\hat{u}=0 \tag{3.2}$$

which has the conserved quantity (energy):

$$E=\frac{1}{2}\hat{u}_\tau\hat{u}_\tau^*+\frac{1}{2}\left(\frac{k^2}{1+k^2}-c^2k^2\right)\hat{u}_\tau\hat{u}_\tau^* \tag{3.3}$$

where $*$ indicates complex conjugate. It is clear that we could not use Eq. 3.3 to prove stability of the $\hat{u}=0$ solution since E is not definite for sufficiently large k. However, it is easily shown that $\hat{u}=0$ is stable by computing the eigenvalues of Eq. 3.1. Therefore, the lack of definiteness of $\delta^2\mathcal{F}$ previously discussed is not indicative of an instability. Nevertheless, the stability of the $u=0$ solution in the moving frame can be deduced by using a different energy functional.

To explore this possibility consider the following system

$$\begin{aligned} \ddot{u} + 2a\dot{v} - u &= 0 \\ \ddot{v} - 2a\dot{u} - v &= 0 \end{aligned} \tag{3.4}$$

which is easily checked to be stable for $a > 1$ and unstable for $a < 1$. This system has the conserved quantity

$$E = \frac{1}{2}(\dot{u}^2 + \dot{v}^2) - \frac{1}{2}(u^2 + v^2) \tag{3.5}$$

which is always indefinite and independent of a. Clearly E offers no hope for proving stability of $u = v = 0$ in Eq. 3.4, but the Hamiltonian does. The Hamiltonian for Eq. 3.4 is

$$\mathcal{H} = \frac{1}{2}(p_u^2 + p_v^2) + \frac{1}{2}(a^2 - 1)(u^2 + v^2) + aQ \tag{3.6}$$

where $Q = vp_u - up_v$. The equations of motion are found in the usual way. Stability follows for $a > 1$ once it is noticed[1] that

$$\frac{dQ}{dt} = 0 \tag{3.7}$$

Let us see what insight this observation has concerning the stability of the solitary wave problem. The equations of motion, (Eq. 2.3), linearized about a solitary wave (in a moving coordinate system, $z = x - ct$) are

$$\begin{pmatrix} v_\tau \\ p_\tau \end{pmatrix} = \begin{pmatrix} c\partial_z & -B^{-1} \\ -\partial_z^2 - \partial_z y^n \partial_z & c\partial_z \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix} \tag{3.8}$$

which has the the conserved quantity

$$\mathcal{H} = \int_{-\infty}^{+\infty} \left(\frac{1}{2} vAv + \frac{1}{2} pB^{-1}p \right) dz + cQ \tag{3.9}$$

where

$$Q = \int_{-\infty}^{+\infty} pv_z dz \quad \text{and} \quad A = -\partial_z^2 - \partial_z y^n \partial_z \tag{3.10}$$

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[1] I thank Robert Pego for this observation.

The Q here is analogous to the one given by Eq. 3.6 and A and B are positive self-ajoint operators. Differentiating Q with respect to time and using Eq. 3.8 we find

$$\dot{Q} = \frac{1}{2} \int_{-\infty}^{+\infty} v[A, \partial_z] v dz \qquad (3.11)$$

where the square brackets denote the commutator of two operators ($[A, B] = AB - BA$). Notice if we have the just trivial solution then $y = 0$ and A commutes with ∂_z so $\dot{Q} = 0$. This is an "energy" proof of stability since A and B are positive. Recall that for this case $\delta^2 \mathcal{F}$ was not definite. For the case of the solitary wave we have

$$\dot{Q} = \frac{n}{2} \int_{-\infty}^{+\infty} v_z^2 y^{n-1} y' dz \qquad (3.12)$$

which is not necessarily zero. However the $y^{n-1}y'$ term is very localized near $x = 0$ and consequently it maybe possible to use Eq. 3.12 to obtain some bounds on Q sufficient to prove stability. Work is in progress in this direction.

4. Numerical Work. Let us briefly present the numerical scheme and some of the previously discussed results. To study Eq. 1.1 numerically we write it in the following form

$$u_{tt} + \frac{u^{n+1}}{n+1} + u = \psi \qquad (4.1)$$

$$\psi_{xx} - \psi = -u - \frac{u^{n+1}}{n+1} \qquad (4.2)$$

Eq. 4.2 is solved as a boundary value problem with $\psi(L) = \psi(-L) = 0$ for L sufficiently large so the boundaries do not interact with the solitary wave. Once ψ is known Eq. 4.1 can be solved as a set of ordinary differential equations. In the implementation of the scheme Eq. 4.2 was center differenced and solved with a tri-diagonal solver and Eq. 4.1 was integrated using a 3rd order Adams-Bashforth scheme.

Numerical calculations show that when Eq. 2.15 is not satisfied the solitary wave appear to be stable. The following figures show the evolution of an unstable solitary wave for $n = 10$ and $c = 1.05$ in a frame moving with speed c. In these computations the solitary wave is perturbed by

$$\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} \varepsilon y \\ 0 \end{pmatrix}$$

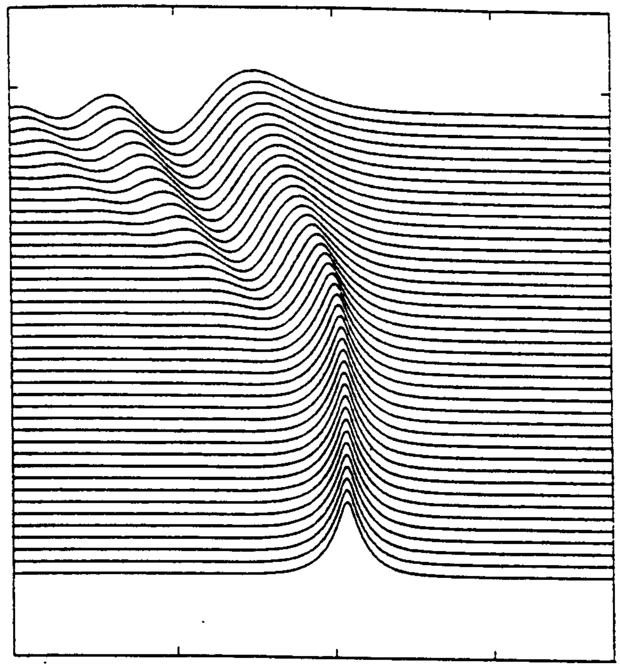

Figure 1. The evolution of an unstable solitary wave. Time increases upwards.

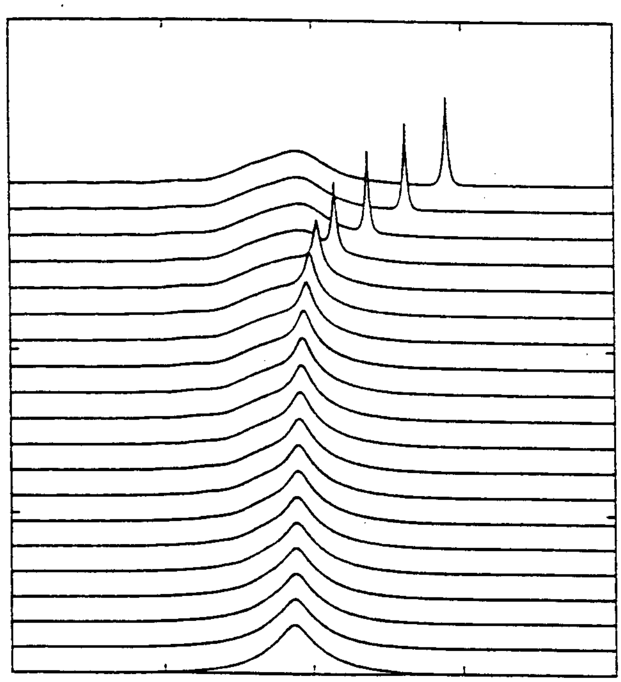

Figure 2. Same as Figure 1 but with different initial conditions. (see text for details)

In Figure 1 $\varepsilon = -.001$ and the solitary wave radiates away. In Figure 2 $\varepsilon = .001$ and we see the solitary wave breaks up into a new stable solitary wave and a radiative component. Eq. 2.15 shows stable solitary waves must travel faster than unstable ones. This behavior is clearly captured in this figure. The work of Pego & Weinstein (1991) combined with the numerical calculation of the Evans function suggest the existence of a one dimensional unstable manifold and these numerical results indicate the behavior is quite different on each branch.

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