

VOID WAVE PROPAGATION IN BUBBLY FLOWS

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Reference:

Wang & Smereka, SIAM J. APPL. MATH. **63** 1849-1888 (2003)

Goals

- The purpose is study void wave propagation in bubbly flows at high Reynolds numbers where potential flow can be considered a good approximation.
- The approach we take is to derive equations motion for individual bubbles and from these formulate effective equations by taking a continuum limit.
- In our analysis we also considered sound propagation but for this presentation it is ignored.

Velocity Potential

For a flow with N disjoint spherical bubbles, we want to find the velocity potential satisfying:

$$\begin{aligned}\Delta\phi &= 0 && \text{outside the bubbles,} \\ \frac{\partial\phi}{\partial n} &= \mathbf{U}_i \cdot \mathbf{n} && \text{on the surface of bubble } i, i = 1 \cdots N, \\ \nabla\phi &= 0 && \text{at infinity,}\end{aligned}$$

where \mathbf{U}_i is the translational velocity of bubble i , \mathbf{n} is the unit normal vector pointing toward the liquid phase on the surface, and

$$\frac{\partial\phi}{\partial n} = \mathbf{n} \cdot \nabla\phi$$

is the directional derivative along \mathbf{n} .

Solution for ϕ

If $N = 1$ then $\phi_1(\mathbf{r}) = \frac{1}{2}R_1^3 \nabla_r \left(\frac{1}{|\mathbf{r}-\mathbf{x}_1|} \right) \cdot \mathbf{U}_1$.

If $N = 2$ then

$$\phi = \phi_1 + \phi_2 + I_1\phi_2 + I_2\phi_1 + I_2I_1\phi_2 + I_1I_2\phi_1 + I_2I_1I_2\phi_1 + I_1I_2I_1\phi_2 + \dots$$

where I_1 and I_2 are the image operators (method of images).

Theorem 1

$$\phi = \sum_{i_1=1}^N \phi_{i_1} + \sum_{\substack{i_1, i_2=1 \\ i_1 \neq i_2}}^N I_{i_1} \phi_{i_2} + \dots + \sum_{\substack{i_1, \dots, i_k=1 \\ i_l \neq i_{l+1}}}^N I_{i_1} I_{i_2} \dots I_{i_{k-1}} \phi_{i_k} + \dots,$$

where I_i is the image operator for bubble i . This is a generalization of the method of images.

Remark For sound propagation a monopole term must be included in ϕ_i .

Ambient Potential

Next we define the ambient velocity potential experienced by the j th bubble to be

$$\psi_j = \sum_{i_1 \neq j}^N \phi_{i_1} + \sum_{\substack{i_1, i_2 = 1 \\ i_1 \neq j, i_1 \neq i_2}}^N I_{i_1} \phi_{i_2} + \cdots + \sum_{\substack{i_1, \dots, i_k = 1 \\ i_1 \neq j, i_\ell \neq i_{\ell+1}}}^N I_{i_1} I_{i_2} \cdots I_{i_{k-1}} \phi_{i_k} + \cdots$$

ψ_j is the part of ϕ that is harmonic inside bubble j .

$\mathbf{v}_j = \nabla \psi_j$ is the ambient liquid velocity.

Two useful expressions:

$$\phi = \phi_j + \psi_j + I_j \psi_j \quad \text{and} \quad \psi_j = \sum_{j \neq k} (\phi_k + I_k \psi_k).$$

The Image Operator

Theorem 2 *If $f(\mathbf{x})$ is harmonic inside a bubble centered at \mathbf{p} with radius R then*

$$\begin{aligned} I_B f(\mathbf{x}) &= \sum_{n=1}^{\infty} \frac{(-1)^n R^{2n+1} \nabla^n f(\mathbf{p}) \cdot \nabla_{\mathbf{x}}^n \left(\frac{1}{|\mathbf{x}-\mathbf{p}|} \right)}{(n-1)!(n+1)(2n-1)!!} \\ &= -\frac{1}{2} R^3 \nabla f(\mathbf{p}) \cdot \nabla_{\mathbf{x}} \left(\frac{1}{|\mathbf{x}-\mathbf{p}|} \right) + (\text{higher order harmonics}) \end{aligned}$$

This means that the leading order effect of one bubble on another is a dipole term.

Euler-Lagrange Equations

The kinetic energy of the liquid is

$$K = \frac{1}{2}\rho_\ell \int_{V_\ell} |\nabla\phi|^2 dv = 2\pi\rho_\ell \sum_{i=1}^N \left(\frac{1}{6}R_i^3 \mathbf{U}_i^2 - \frac{1}{2}R_i^3 \mathbf{U}_i \cdot \mathbf{v}_i(\mathbf{x}_i) \right)$$

where \mathbf{v}_i is the ambient velocity.

The Euler-Lagrange equations are: $\frac{d}{dt} \frac{\partial K}{\partial \mathbf{U}_i} - \frac{\partial K}{\partial \mathbf{x}_i} = 0$

One can show the following (a long difficult calculation)

$$\frac{\partial K}{\partial \mathbf{U}_i} = 2\pi\rho_\ell \left(\frac{1}{3}R_i^3 \mathbf{U}_i - R_i^3 \mathbf{v}_i \right)$$

$$\frac{\partial K}{\partial \mathbf{x}_i} = 2\pi\rho_\ell \left(-2R_i^2 \dot{R}_i \mathbf{v}_i + R_i^3 (\nabla \mathbf{v}_i)^T \cdot (\mathbf{v}_i - \mathbf{U}_i) + G \right)$$

where G only involves spherical harmonics of higher order than a dipole.

Ambient Velocity

$$\mathbf{v}_i = \nabla \psi_i \quad \text{where} \quad \psi_i = \sum_{i \neq k} (\phi_k(\mathbf{x}_i) + I_k \psi_k(\mathbf{x}_i))$$

Simplifying approximation: keep only dipoles.

From Theorem 2 we have:

$$I_k \psi_k(\mathbf{x}_i) = -\frac{1}{2} R_k^3 \mathbf{v}_k \cdot \nabla_{\mathbf{x}_k} \left(\frac{1}{|\mathbf{x}_i - \mathbf{x}_k|} \right) + (\text{higher order harmonics}).$$

Combining the two formulas we have:

$$\psi_i = \sum_{k \neq i} \frac{1}{2} R_k^3 \nabla_{\mathbf{x}_i} \left(\frac{1}{|\mathbf{x}_i - \mathbf{x}_k|} \right) \cdot (\mathbf{U}_k - \mathbf{v}_k),$$

Equations of Motion

$$\frac{1}{3}\dot{\mathbf{U}}_i - \frac{D\mathbf{v}_i}{Dt} = (\mathbf{v}_i - \mathbf{U}_i) \times (\nabla \times \mathbf{v}_i)$$

$$\mathbf{v}_i = \nabla_{\mathbf{x}_i} \psi_i$$

$$\psi_i = \sum_{k \neq i} \frac{1}{2} R_k^3 \nabla_{\mathbf{x}_i} \left(\frac{1}{|\mathbf{x}_i - \mathbf{x}_k|} \right) \cdot (\mathbf{U}_k - \mathbf{v}_k)$$

$$D/Dt = \partial_t + \mathbf{v} \cdot \nabla$$

Continuum Limit: An example

Consider N point charges located at \mathbf{x}_j with charge q_j where $j = 1, \dots, N$. The ambient electric potential at \mathbf{x}_i is

$$\psi_i(\mathbf{x}_i) = \sum_{j \neq i}^N \frac{q_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

We suppose that there exists a function $q(\mathbf{x})$ such that $q_j = q(\mathbf{x}_j)$ then this summation can be approximated as

$$\psi(\mathbf{x}) = \int \frac{\rho(\mathbf{y})q(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}$$

where $\rho(\mathbf{x})$ is the number of particles per unit volume. This is equivalent to

$$\Delta\psi = -4\pi q(\mathbf{x})\rho(\mathbf{x})$$

Continuum Limit

Assume: that there exists a function $\mathbf{U}(\mathbf{x}, t)$ such that $\mathbf{U}_k = \mathbf{U}(\mathbf{x}_k, t)$ (No velocity fluctuations, “Cold Bubbly Flows”)

The continuum limit for ψ is $\Delta\psi = -\frac{3}{2}\nabla \cdot (\beta(\mathbf{U} - \mathbf{v}))$

We also find: $\mathbf{v} = \nabla\psi + \frac{\beta}{2}(\mathbf{U} - \mathbf{v})$

β is the void fraction and it satisfies

$$\frac{\partial\beta}{\partial t} + \nabla \cdot (\beta\mathbf{U}) = 0$$

One can show $\nabla \cdot (\beta\mathbf{U} + (1 - \beta)\mathbf{v}) = 0$

This is exactly the conservation of total volume. This indicates that to the level of our approximation that \mathbf{v} is also the volume averaged liquid velocity.

Void Waves

Equations of Motion in 1D are:

$$\frac{1}{3} \left(\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \right) - \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) = 0,$$

$$\frac{\partial \beta}{\partial t} + \frac{\partial}{\partial x} (\beta U) = 0$$

$$(1 - \beta)v + \beta U = 0$$

Next we consider a new variable

$$M = \beta U / h(\beta) \quad \text{where} \quad h(\beta) = 2\beta(1 - \beta) / (1 + 2\beta)$$

Effective equations are the following conservation laws:

$$\frac{\partial \beta}{\partial t} + \frac{\partial}{\partial x} (h(\beta)M) = 0,$$

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial x} \left(\frac{1}{2} h'(\beta) M^2 \right) = 0.$$

Previous Work

These equations are the same as those derived by Geurst, Wallis, Pauchon & Smereka provided we use Zuber's expression for the virtual mass.

$$m(\beta) = \rho_l \frac{\beta}{2} \left(\frac{1 + 2\beta}{1 - \beta} \right).$$

As pointed out Pauchon & Smereka these equations are ill-posed. Russo & Smereka showed that velocity fluctuations can make them well-posed.

Effects of Liquid Viscosity and Gravity

The Lagrangian is given by $L = K - U_g$ where U_g is the potential energy due gravity.

We shall include effects of liquid viscosity by using a dissipation function denoted as D . The equations of motion are now:

$$\frac{d}{dt} \frac{\partial L}{\partial \mathbf{U}_i} - \frac{\partial L}{\partial \mathbf{x}_i} = \frac{1}{2} \frac{\partial D}{\partial \mathbf{U}_i}$$

The amount of energy dissipated is given by:

$$D = 2\mu \int_{V_\ell} \varepsilon_{ij} \varepsilon_{ij} d\mathbf{x} \quad \text{where} \quad \varepsilon_{ij} \quad \text{is the rate of strain tensor.}$$

High Reynolds Number Approximation

We shall assume the Reynolds number is high, so the flow is close to potential flow except in the thin boundary layer wrapped around each bubble. We shall further assume that no significant amount of energy dissipates in the boundary layer. This assumption for one bubble was justified by Moore(1963). Therefore

$$D = -\mu \int_S \frac{\partial(\nabla\phi \cdot \nabla\phi)}{\partial n} ds,$$

where the integral is taken on the surface of the bubbles, and \mathbf{n} is outward normal vector.

Calculation of Drag - Single Bubble

We consider a single bubble moving with a fixed radius and a translational velocity, \mathbf{U} , in a fluid with a constant ambient velocity \mathbf{v}_∞ . The velocity potential in this case is

$$\phi = \frac{1}{2}R^3\nabla_r \left(\frac{1}{|\mathbf{r} - \mathbf{x}|} \right) \cdot (\mathbf{U} - \mathbf{v}_\infty) + \mathbf{v}_\infty \cdot \mathbf{r}.$$

The energy dissipation is computed using the previous formula and found to be

$$\mathcal{D} = 12\pi\mu R|\mathbf{U} - \mathbf{v}_\infty|^2.$$

The drag on the bubble is then given by

$$\mathbf{F} = \frac{1}{2} \frac{\partial \mathcal{D}}{\partial \mathbf{U}} = 12\pi\mu R(\mathbf{U} - \mathbf{v}_\infty)$$

Levich first derived this using the method outlined here.

Calculation of Drag - N Bubbles

Using the approximation for ϕ we find

$$D \approx 12\pi\mu R \sum_{i=1}^N |\mathbf{U}_i - \mathbf{v}_i(\mathbf{x}_i)|^2$$

This compares closely with the one bubble computation with \mathbf{v}_∞ replaced by the ambient velocity. Nevertheless the drag force will be different. A computation shows the continuum limit of the drag force is, approximately,

$$12\pi\mu R(\mathbf{U} - 2\mathbf{v}) = 12\pi\mu R U \frac{1 + \beta + \beta^2}{1 - \beta} \quad \text{in 1D}$$

where \mathbf{v} is the ambient liquid velocity.

Equations of Motion

With the drag force computed we can now modify our model for void waves to include gravity and liquid viscosity. We find:

$$\frac{1}{3} \left(\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \right) - \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) = \frac{2}{3} g \left(1 - \frac{U}{U_\infty} \frac{1 + \beta + \beta^2}{1 - \beta} \right).$$
$$\frac{\partial \beta}{\partial t} + \frac{\partial (\beta U)}{\partial x} = 0$$

where $(1 - \beta)v + \beta U = 0$ and

$$U_\infty = \frac{R^2 \rho_l g}{9\mu}$$

is the steady speed of a single bubble rising in an infinite fluid under the force of gravity.

Steady Solutions

It is easy to verify that the previous equations have the following equilibrium solution: $\beta = \beta_0$ and $U = U_0$, where

$$U_0 = \frac{1 - \beta_0}{1 + \beta_0 + \beta_0^2} U_\infty.$$

This corresponds to a spatially uniform mixture of bubbles rising due to gravity in the zero volume flux frame of reference. The rise speed of the bubbles is given by U_0 . The prediction is in good agreement with the experimental data reported by Lammers & Biesheuvel.

Void Waves

We examine perturbations of the steady solution.

We let $U = U_0(\beta_0) + U'$ and $\beta = \beta_0 + \beta'$ where $U', \beta' \ll 1$ and $U', \beta' \propto e^{i(kx - \omega t)}$.

One finds the problem is still ill-posed. But for small k we find the following phase velocities

$$c_1 \approx U_\infty + i \left(\frac{2g}{kU_\infty} + \frac{3U_\infty^3 k \beta_0}{2g} \right)$$
$$c_2 \approx c_R - i \frac{3U_\infty^3 k \beta_0}{2g}$$

where

$$c_R = \frac{1 - 2\beta_0 - 2\beta_0^2}{(1 + \beta_0 + \beta_0^2)^2} U_\infty$$

Remarks on the phase speeds

- In the expression for c_1 we observe that its imaginary part is positive. This indicates that this mode decays. This corresponds to the relaxation of the bubble's speed to the equilibrium speed.
- The second mode corresponds to void waves. It predicts that the void waves will move with speed c_R and will grow with a rate given by $c_I k$.
- Remarkably, a plot of c_R as function of the volume fraction along with experimental data shows good agreement

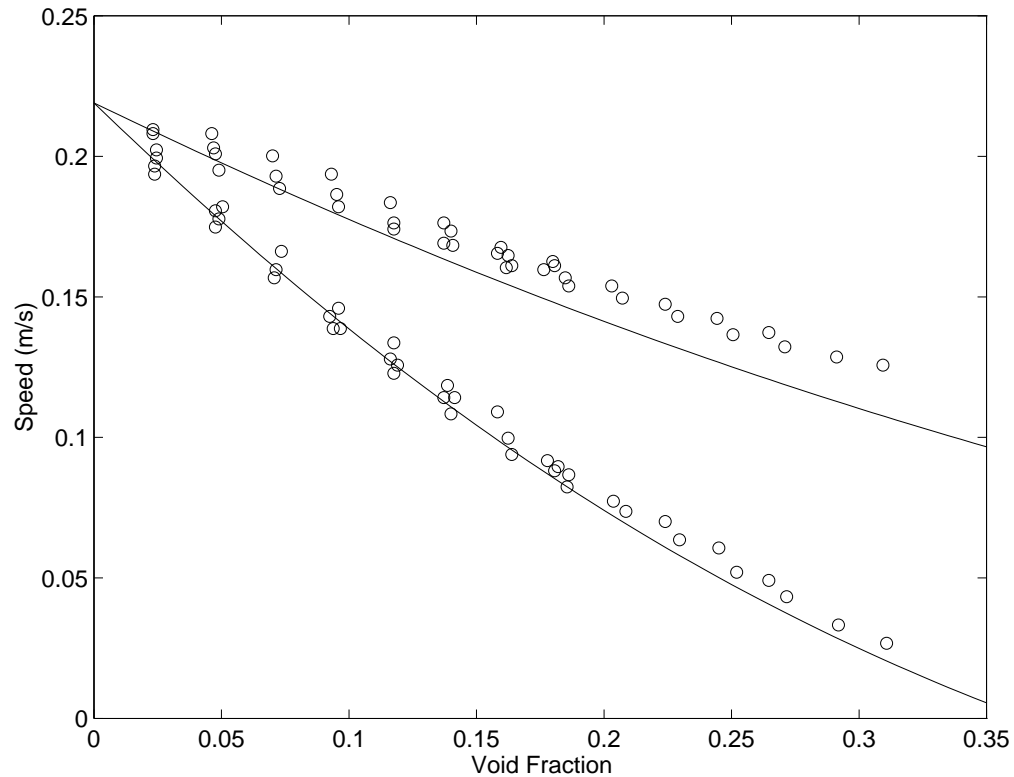


Figure 1: The upper curve shows a plot of the predicted bubble rise speed and the lower curve shows the predicted void wave speed. The circles are the experimental findings of Lammers & Biesheuvel (1996).

Remarks

- The fact the questions are ill-posed is consistent with the bubbly clustering observed in simulations by Sangani & Didwania and Smereka
- Lammers & Biesheuvel report $U_\infty \approx 25$ cm/sec. The frequency of naturally occurring void waves was approximately 1 Hz (see Biesheuvel & Gorissen)
- We can estimate the growth rate to be $\approx \beta$
- This suggests that the void waves do not have time to grow significantly in the normal experimental settings
- Recent experiments of Zenit et al observe a small amount of clustering

- Therefore, it appears that our model provides an accurate description of long wavelength disturbances. It is possible that the model breaks down for small wavelength disturbances and there are regularizing effects which when included in our model will result in a well-posed model
- It also possible that Sangani & Didwania and Smereka over estimate the amount of cluster formation observed in experiments. In these simulations the computational domain was a cube with a size of only a few centimeters thereby exciting modes of a much smaller wavelength than those observed in experiments

Summary and Conclusions

- We have developed a new method for solving Laplace's equation for the velocity potential in a liquid with a finite number of bubbles by generalizing of the method of images
- Our approach also allows us to define the ambient velocity and ambient pressure
- The velocity potential is then used to calculate the total kinetic energy of the liquid. We then use the Euler-Lagrange equation to compute exact equations of motion
- Using the simplifying approximation: keep only terms arising from monopoles and dipoles we obtain a tractable set equations
- We then take the continuum limit of the equations of motion to obtain a set of partial differential equations which represent our effective equations for ideal bubbly fluids. Our model includes both sound and void wave propagation

- We show that our model captures the results of the speed of sound waves by Caflisch *et al*, Crespo, and Sangani. We also show that our model reduces to Geurst's model when we consider void wave propagation
- The drag force is computed using a energy dissipation function and we are able to compute rise speed of a mixture of bubbles which is in good agreement with experiments
- We also compute the speed of void waves and find good agreement with experiments
- It remains to systematically develop a set of well-posed equations that faithfully describe the experiments.