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EFFECTIVE EQUATIONS FOR SOUND AND VOID WAVE PROPAGATION IN BUBBLY FLUIDS*

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Abstract. Effective equations that describe both sound wave and void wave propagation for bubbly flows at high Reynolds numbers are derived in this paper. First ideal bubble flows are considered, and a new method for solving Laplace's equation for the velocity potential is presented. This approach is based on a generalization of the method of images and also yields a precise definition of the ambient field experienced by a bubble. With the velocity potential known, the Lagrangian is then computed, and equations of motion for a finite number of bubbles using the Euler–Lagrange equations are derived. The continuum limit is then used to obtain our effective equations. Our expressions for the sound wave and void wave speeds agree well with previous investigations. The effects of gravity and viscosity on void waves are considered. Viscous effects are incorporated using a dissipation function. The steady rise speed and void wave speed for a column of rising bubbles are computed and found to agree well with experiments.

Key words. bubbly flow, potential flow, void waves

AMS subject classifications. 76T10, 76B07

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1. Introduction.

In this paper, we derive effective equations for sound and void wave propagation in bubbly fluids. Sound propagation was studied by Carstensen and Foldy [11], who derived the speed of sound using a linear scattering theory developed by Foldy [13]. Iordanskii [21] and van Wijngaarden [48] derived effective equations including nonlinear effects. For review of the literature on acoustic waves in bubbly liquids the reader is referred to the article by van Wijngaarden [49]. Later, Caflisch et al. [9] provided an alternate derivation that clarified the range of validity of the effective equations derived by Iordanskii and van Wijngaarden. These equations are valid when the volume fraction of bubbles is very small. This is, in part, because in these investigations it was assumed that bubbles would undergo only radial motion. In order to develop equations valid at higher volume fraction, one must include the effects of bubble translation. This has been investigated by Crespo [12], Noordzij and van Wijngaarden [32], Caflisch et al. [10], and Sangani [38], among others. Crespo used volume averaged equations of motion, which are valid for low frequency perturbations. He then linearized these equations of motion and computed the speed of sound waves, the results of which were found to be in good agreement with experiments. Caflisch et al. [10] linearized the equations of motion and used a multiple-scale method. Their computation of the wave speed is valid only for small frequencies and was in agreement with the results of Crespo. Sangani also linearized the equations of motion and then performed ensemble averaging. He also computed the wave speed, and his expression was valid over a wide range of wave frequencies. Sangani's results compare well to the experimental results of Silberman [40]. In this work we shall derive equations of motion that are fully nonlinear and valid over a wide range of frequencies. Our

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calculation of the sound speed agrees with those of previous investigators.

There has been considerable work on void wave propagation for ideal bubbly flows where the bubbles are considered to be rigid spheres surrounded by an incompressible, inviscid, irrotational fluid. For example, Biesheuvel and van Wijngaarden [8], Geurst [17, 18], Biesheuvel and Gorissen [6], Wallis [47], Pauchon and Smereka [34], Zhang and Prosperetti [56], and Park, Drew, and Lahey [33], among others, have derived effective equations using various types of averaging. The motion of individual bubbles was studied numerically by Sangani and Didwania [39] and Smereka [42]. Smereka used the point bubble approximation combined with Euler–Lagrange equations to obtain explicit equations of motion which were integrated numerically. Sangani and Didwania used a multipole method together with Newton's law to simulate bubble motion. Both investigations observed that in many situations a spatially uniform mixture of bubbles moving with approximately the same velocity was unstable and the bubbles would form clusters. In should also be mentioned that van Wijngaarden [53] predicted that bubbly flows would have a tendency to cluster. We speculate that the instability of the spatially uniform mixture of bubbles is consistent with the ill-posed effective equations found by Geurst [17], Wallis [47], and Pauchon and Smereka [34] for dilute bubbly flows.

It was also found by Sangani and Didwania [39] and Smereka [42] that, if gravity and viscosity were included, the clustering was much more pronounced. Smereka constructed a Lyapunov function and showed that bubbles have a strong tendency to maximize their virtual mass in the direction of motion. This means that the bubbles will form pancake-shaped clusters. The dynamics of clustering has been studied in more detail by Yurkovetsky and Brady [55] and Galper and Miloh [16]. Many aspects of bubbly flows with potential flow interaction have also been discussed in the review article by Koch and Hill [24].

In an effort to understand the numerical simulations of Sangani and Didwania and Smereka, Russo and Smereka [37] wrote the equations of motion for individual bubbles in Hamiltonian form (using the point bubble approximation) and then deduced a kinetic equation for the probability density of the bubbles in phase space. They proved that the spatially uniform case was unstable, provided that the variance of the bubble's velocity was sufficiently small. On the other hand, they proved that, if the variance of the bubble's velocity was sufficiently large, the spatially uniform bubbly fluid was stable. Similar results were obtained by Spelt and Sangani [45]. Herrero, Lucquin-Desreux, and Perthame [20] were able to provide a more rigorous derivation of the equations derived by Russo and Smereka and remove an important assumption.

The effects of liquid viscosity have also been considered by van Wijngaarden and Kapteyn [52] and van Wijngaarden [53]. Van Wijngaarden and Kaptyen computed the drag force on a pair of bubbles using an energy dissipation argument. They then computed averaged equations using ensemble averages over pairs of bubbles. The results were used to compute the profile of a wave of steady shape. Van Wijngaarden used the results of van Wijngaarden and Kapteyn to compute the rise speed of a mixture of bubbles. The result is in good agreement with experimental data.

More recently, Lammers and Biesheuvel [26] studied void waves in bubbly flows using a theory of Batchelor [4]. They find that the speed c of void waves is given by

$$c = U_0(\beta) + \beta U_0'(\beta), \tag{1}$$

where U_0 is the rise speed of the bubbles and β is the void fraction. They measure both c and U_0 and find that they agree well with (1). In the current work we also obtain (1) by a different approach. In addition, we calculate U_0 and find that it agrees closely with the experimental results of Lammers and Biesheuvel.

2. Outline. We shall derive effective equations by first computing the equations of motion of N bubbles surrounded by an ideal liquid of infinite extent. We assume that the bubbles are spherical but may change their radius. To fix ideas let us first consider a single spherical gas bubble in a liquid which is at rest at infinity. The equations of motion are well known; they are

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 - \frac{1}{4}|\mathbf{U}|^2 + \frac{p_\infty - P}{\rho_\ell} = 0, \tag{2}$$

$$\frac{1}{3}\dot{\mathbf{U}} + \frac{\dot{R}}{R}\mathbf{U} = 0, \tag{3}$$

where $R(t)$ is the bubble radius, $\mathbf{U}(t)$ is the bubble velocity, ρ_ℓ is the density of the liquid, p_∞ is the pressure at infinity, and P is the pressure inside the bubble. Surface tension is neglected for the purpose of simplicity. Equation (2) with $\mathbf{U} = 0$ can be found in Lamb [25], for example. The inclusion of (3) can be found in Hermans [19], for example.

Next we consider the situation in which the surrounding liquid is uniformly accelerated. The equations of motion are

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 - \frac{1}{4}|\mathbf{U} - \mathbf{v}|^2 + \frac{p_\infty - p_g}{\rho_\ell} = 0, \tag{4}$$

$$\frac{1}{3}\dot{\mathbf{U}} - \dot{\mathbf{v}} + \frac{\dot{R}}{R}(\mathbf{U} - \mathbf{v}) = 0, \tag{5}$$

where $\mathbf{v}$ is the velocity of the liquid at infinity. The equations of motion in this case are derived by first obtaining the pressure at the bubble surface from Bernoulli's law and then demanding that the average pressure on the surface be equal to the pressure inside the bubble and that total force on the bubble be zero. Equation (5) can be found in Batchelor [3, p. 455].

Let us consider the situation with N bubbles surrounded by a fluid of infinite extent initially at rest. The bubbles are then set into motion. It is plausible to think that each bubble moves only according to certain "ambient" fields. Therefore we write the equations of motion, heuristically, for the kth bubble as

$$R_k\ddot{R}_k + \frac{3}{2}\dot{R}^2 - \frac{1}{4}|\mathbf{U}_k - \mathbf{v}_A(k)|^2 + \frac{p_A(k) - P_k}{\rho_\ell} = 0, \tag{6}$$

$$\frac{1}{3}\dot{\mathbf{U}}_k - \frac{D_?}{Dt}\mathbf{v}_A(k) + \frac{\dot{R}_k}{R_k}(\mathbf{U}_k - \mathbf{v}_A(k)) = 0, \tag{7}$$

where $\mathbf{v}_A(k)$ and $p_A(k)$ are the ambient liquid velocity and the ambient pressure of the kth bubble, which must be determined. $\frac{D_?}{Dt}$ denotes a material derivative associated to a velocity field yet to be determined. One of the key results of this paper is a systematic way to determine these ambient fields.

2.1. Summary and approach. Our approach is as follows. First consider a finite number of bubbles in an infinite expanse of fluid. We assume that the fluid motion is irrotational, inviscid, incompressible, and at rest at infinity. We also assume that the bubbles are spherical. We then derive equations of motion for this finite collection of bubbles using Lagrange's variational principal as outlined in Lamb [25] or Milne-Thompson [30], for example. This requires one to compute the velocity potential. We develop a new method to solve for the velocity potential. The method is an extension of the method of images for two bubbles (e.g., Lamb [25]) to multiple bubbles. We prove that this method is convergent.

With the velocity potential known, we can calculate the Lagrangian for a finite number of bubbles. In principle we could calculate the exact equations of motion; however, they would be extremely complex. Instead we truncate our equations of motion and include only terms involving monopoles and dipoles. In addition, we are able to systematically deduce the ambient fields. We then take the continuum limit of our discrete equations of motion and find the following effective equations:

$$R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2 - \frac{1}{4}|\mathbf{U}-\mathbf{v}|^2 + \frac{p - p_g(R)}{\rho_\ell} = 0, \tag{8}$$

$$\frac{1}{3}\frac{d\mathbf{U}}{dt} - \frac{D\mathbf{v}}{Dt} + \frac{1}{R}\frac{dR}{dt}(\mathbf{U}-\mathbf{v}) + (\mathbf{U}-\mathbf{v}) \times (\nabla \times \mathbf{v}) = 0, \tag{9}$$

where $\frac{d}{dt} = \partial_t + \mathbf{U}\cdot\nabla$ and $\frac{D}{Dt} = \partial_t + \mathbf{v}\cdot\nabla$. The dependent variables are now functions of space and time (e.g., $R = R(\mathbf{x}, t)$). The ambient pressure p and ambient liquid velocity $\mathbf{v}$ are related as follows:

$$\mathbf{v} = \nabla\psi \quad \text{and} \quad p = p_\infty - \frac{\partial\psi}{\partial t} - \frac{1}{2}|\mathbf{v}|^2,$$

where p_∞ is the pressure at infinity and ψ is the ambient velocity potential. An explicit expression for ψ is given by (42). We also establish that, to leading order, $\mathbf{v}$ is the volume averaged liquid velocity.

We also consider the effects of gravity and liquid viscosity. We include the viscous effects by using the energy dissipation method. Our approach is similar to that of van Wijngaarden and Kapteyn [52] and van Wijngaarden [53] except that we do not use the assumption of pairwise interaction. When we pass to the continuum limit we find that, to leading order, the drag force on a bubble, in the case of zero volume flux, is

$$12\pi\mu R(\mathbf{U} - 2\mathbf{v} - \mathbf{w}).$$

The expression for $\mathbf{w}$ is given by (79) in section 5. The above formula in one space dimension can be written as

$$12\pi\mu R(1 + \beta + \beta^2)(U - v).$$

This formula is also valid for all cases when the volume flux is constant in time.

We include this formula in our model along with effects of gravity to study the propagation of void waves in bubbly flow. Work of Sangani and Didwania [39] and Smereka [42] suggest that the potential flow approximation cannot be used for void wave propagation in bubbly flows since it predicts strong clustering of the bubbles, which is something not observed in experiments. Recent experiments of Zenit, Koch, and Sangani [58] show that there is some clustering but not to the extent predicted in [39, 42].

We show that our model has a steady solution which corresponds to a spatially homogeneous bubble mixture; the computed velocity is in good agreement with experiments. Furthermore, we demonstrate that this steady solution is unstable, which is in agreement with [39, 42]. We then argue that for naturally occurring perturbations the instability is rather weak. The propagation of these perturbations corresponds to void waves. We calculate the speed of these waves and find that our calculations agree closely with experimental results.

3. Equations of motion. We neglect liquid viscosity and gravity in this chapter. The total energy, the sum of the kinetic energy of the liquid and potential energy stored in the bubbles, is conserved. The Lagrangian is calculated, and the Euler–Lagrange equations give the equations of motion. The velocity potential, given as a convergent series, and its combinatorial properties play an important role in the derivation.

3.1. Velocity potential. For a flow with N disjoint spherical bubbles, we want to find the velocity potential satisfying

$$\triangle\phi = 0 \quad \text{outside the bubbles,} \tag{10}$$

$$\frac{\partial \phi}{\partial n} = \mathbf{U}_i \cdot \mathbf{n} + \dot{R}_i \quad \text{on the surface of bubble } i,\ i = 1, \ldots, N, \tag{11}$$

$$\nabla\phi = 0 \quad \text{at infinity,} \tag{12}$$

where $\dot{R}_i$, $\mathbf{U}_i$ are radial and translational velocities of bubble i, $\mathbf{n}$ is the unit normal vector pointing toward the liquid phase on the surface, and

$$\frac{\partial \phi}{\partial n} = \mathbf{n} \cdot \nabla\phi$$

is the directional derivative along $\mathbf{n}$. We have the following result concerning the solution of (10)–(12).

THEOREM 3.1. *Let*

$$\phi_i(\mathbf{r}) = -\frac{R_i^2 \dot{R}_i}{|\mathbf{r} - \mathbf{x}_i|} + \frac{1}{2} R_i^3 \nabla_r \left(\frac{1}{|\mathbf{r} - \mathbf{x}_i|} \right) \cdot \mathbf{U}_i. \tag{13}$$

Here ϕ_i is the solution if only the ith bubble is present. The solution of (10)–(12) *can be written as the uniformly convergent series in the liquid,*

$$\phi = \sum_{i_1=1}^{N} \phi_{i_1} + \sum_{\substack{i_1, i_2 = 1 \\ i_1 \neq i_2}}^{N} I_{i_1} \phi_{i_2} + \cdots + \sum_{\substack{i_1, \ldots, i_k = 1 \\ i_\ell \neq i_{\ell+1}}}^{N} I_{i_1} I_{i_2} \cdots I_{i_{k-1}} \phi_{i_k} + \cdots, \tag{14}$$

where I_i refers to the image potential operator with respect to bubble i. I_i is defined as follows: let $f(\mathbf{x})$ be a harmonic function inside the ith bubble and let $g(\mathbf{x})$ be a harmonic function outside the ith bubble such that

$$\frac{\partial f}{\partial n} = -\frac{\partial g}{\partial n} \tag{15}$$

on the surface of the ith bubble; then g is called the image potential of f with respect to bubble i with the notation $g = I_i f$.

We note that this is a generalization of the method of images used to solve the motion of two spheres as outlined in Lamb [25].

Next we define the ambient velocity potential experienced by the jth bubble to be

$$\psi_j = \sum_{i_1 \neq j}^{N} \phi_{i_1} + \sum_{\substack{i_1, i_2 = 1 \\ i_1 \neq j, i_1 \neq i_2}}^{N} I_{i_1} \phi_{i_2} + \cdots + \sum_{\substack{i_1, \ldots, i_k = 1 \\ i_1 \neq j, i_\ell \neq i_{\ell+1}}}^{N} I_{i_1} I_{i_2} \cdots I_{i_{k-1}} \phi_{i_k} + \cdots. \tag{16}$$

The ambient liquid velocity experienced by bubble j is defined as $\mathbf{v}_j = \nabla\psi_j$. In the expression for ψ_j we see that the final image reflection of each term is not with respect to bubble j. This means that ψ_j is harmonic inside bubble j and $I_j\psi_j$ is well defined.

It is easy to derive two useful expressions,

$$\phi = \phi_j + \psi_j + I_j\psi_j \tag{17}$$

and

$$\psi_j = \sum_{j\neq k} \left(\phi_k + I_k\psi_k\right). \tag{18}$$

Most of the proof of Theorem 3.1 will be in Appendix A. Here we only outline the proof of convergence and show that ϕ satisfies the boundary condition (11). To prove the latter, we first notice that

$$\frac{\partial\phi_i}{\partial n} = \mathbf{U}_i\cdot\mathbf{n} + \dot{R}_i$$

on the surface of bubble i. Thus ϕ_i is the exact solution of (10)–(12) for one bubble. In the case of multiple bubbles, it follows from (17) that

$$\frac{\partial\phi}{\partial n} = \frac{\partial\phi_i}{\partial n} + \frac{\partial\psi_i}{\partial n} + \frac{\partial I_i\psi_i}{\partial n}.$$

From the definition of the operator I, we have

$$\frac{\partial\psi_i}{\partial n} = -\frac{\partial I_i\psi_i}{\partial n}$$

at the surface of the bubble i. Hence we find

$$\frac{\partial\phi}{\partial n} = \frac{\partial\phi_i}{\partial n} = \mathbf{U}_i\cdot\mathbf{n} + \dot{R}_i.$$

To prove convergence we first separate the series into N subseries (one for each bubble), each of which is harmonic in the region exterior to the corresponding bubble. We then prove that this series converges in the energy norm. This allows us to prove that the velocity potential converges on the surface of the corresponding bubble. This enables us to find the velocity potential on the bubble surface. The Poisson kernel for the exterior Dirichlet problem is used to prove that the series converges at each point in the liquid to a solution of Laplace's equation. The details can be found in Appendix A.

3.1.1. Example. To understand some of the preceding formulas more easily it is useful to write them explicitly for the case $N = 2$. We begin with

$$\phi = \phi_1 + \phi_2 + I_1\phi_2 + I_2\phi_1 + I_2I_1\phi_2 + I_1I_2\phi_1 + I_2I_1I_2\phi_1 + I_1I_2I_1\phi_2 + \cdots.$$

The ambient velocity potential for the first bubble is

$$\psi_1 = \phi_2 + I_2\phi_1 + I_2I_1\phi_2 + I_2I_1I_2\phi_1 + \cdots.$$

3.1.2. The image operator. The following theorem provides an explicit formula for the image potential with respect to a bubble centered at $\mathbf{x} = \mathbf{p}$ with radius R (denoted B).

THEOREM 3.2. *If $f(\mathbf{x})$ is harmonic inside a bubble centered at $\mathbf{p}$ with radius R, then*

$$\begin{aligned} I_B f(\mathbf{x}) &= \sum_{n=1}^{\infty} \frac{(-1)^n R^{2n+1} \nabla^n f(\mathbf{p}) \cdot \nabla_{\mathbf{x}}^n (\frac{1}{|\mathbf{x}-\mathbf{p}|})}{(n-1)!(n+1)(2n-1)!!} \\ &= -\frac{1}{2} R^3 \nabla f(\mathbf{p}) \cdot \nabla_{\mathbf{x}} \left(\frac{1}{|\mathbf{x}-\mathbf{p}|} \right) + \textit{(higher order harmonics)}. \end{aligned}$$

The proof is given in Appendix B.

This theorem expresses the image potential as an expansion of spherical harmonics centered at $\mathbf{p}$. With this theorem, (14) can be written as a multipole series, and it becomes a natural extension of the twin spherical expansion method, which is commonly used when solving Laplace's equation outside two spheres (e.g., Ross [36] or Jeffrey [22]). ∇^n is the nth order matrix of partial derivatives, i.e.,

$$\nabla^n = \frac{\partial^n}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_n}},$$

where $k_j = 1, 2,$ or 3, with $j = 1, \ldots, n$. $\nabla^n f \cdot \nabla^n g$ denotes the scalar product of two nth order matrices.

3.2. Kinetic energy and potential energy. The kinetic energy of the liquid is

$$K = \frac{1}{2}\rho_\ell \int_{V_\ell} |\nabla\phi|^2 dv = \frac{1}{2}\rho_\ell \int_{V_\ell} \nabla\cdot(\phi\nabla\phi) dv = -\frac{1}{2}\rho_\ell \sum_{j=1}^{N} \int_{S_j} \phi \frac{\partial \phi}{\partial n} ds, \tag{19}$$

where ρ_ℓ is the density of the liquid, V_ℓ is the volume occupied by the liquid, and S_i is the surface of the ith bubble. From (17), we have

$$K = -\frac{1}{2}\rho_\ell \sum_{j=1}^{N} \int_{S_j} (\phi_j + \psi_j + I_j\psi_j)(\dot{R}_j + \mathbf{U}_j \cdot n) ds.$$

Substituting (13) into the above expression, we obtain

$$K = -\frac{1}{2}\rho_\ell \sum_{j=1}^{N} \int_{S_j} \left(-R_j\dot{R}_j - \frac{1}{2} R_j \mathbf{U}_j \cdot \mathbf{n} + \psi_j + I_j\psi_j \right) \cdot (R_j + \mathbf{U}_j \cdot \mathbf{n}) ds.$$

Next, we use the expressions in Appendix C, and the kinetic energy of the liquid becomes

$$K = 2\pi\rho_\ell \sum_{i=1}^{N} \left(R_i^3 \dot{R}^2 + \frac{1}{6} R_i^3 \mathbf{U}_i^2 - R_i^2 \dot{R}_i \psi_i(\mathbf{x}_i) - \frac{1}{2} R_i^3 \mathbf{U}_i \cdot \mathbf{v}_i(\mathbf{x}_i) \right). \tag{20}$$

The first two terms in the parentheses correspond to the energy generated by the motion of each individual bubble, as if no other bubbles exist. The last two terms are caused by the interactions between the bubbles through the ambient fields ψ_i and $\mathbf{v}_i$.

We assume that there is no heat transfer involved in the radial oscillation of the bubbles and that the adiabatic constant for the gas is γ. Therefore the pressure inside the bubble is

$$p_g(R) = p_\infty \left(\frac{\rho_0}{\rho}\right)^\gamma = p_\infty \left(\frac{R_0}{R}\right)^{3\gamma}.$$

Therefore the potential energy for a single bubble is

$$\begin{aligned}
-\int_{R_0}^{R} 4\pi R^2 (p_g(R) - p_\infty) dR &= -\int_{R_0}^{R} 4\pi R^2 p_\infty \left(\left(\frac{R_0}{R}\right)^{3\gamma} - 1\right) dR \\
&= 4\pi p_\infty \left(\frac{R_0^{3\gamma} R^{-3\gamma+3}}{3\gamma - 3} + \frac{1}{3}R^3 - \frac{R_0^3}{3\gamma-3} - \frac{1}{3}R_0^3\right).
\end{aligned}$$

The total potential energy of the bubbles is

$$\mathcal{U}_g = 4\pi p_\infty \sum_{i=1}^{N} \left(\frac{R_0^{3\gamma} R_i^{-3\gamma+3}}{3\gamma - 3} + \frac{1}{3}R_i^3 - \frac{R_0^3}{3\gamma-3} - \frac{1}{3}R_0^3\right). \tag{21}$$

We remark that the assumption that the bubbles behave adiabatically can be removed by using the approach outlined by Smereka [44].

3.3. Euler–Lagrange equations. With the kinetic and potential energy obtained in the last two sections, we can write the Lagrangian of the system as

$$\mathcal{L} = \mathcal{K} - \mathcal{U}_g.$$

The Euler–Lagrange equations are

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{R}_i} - \frac{\partial \mathcal{L}}{\partial R_i} &= 0, \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \mathbf{U}_i} - \frac{\partial \mathcal{L}}{\partial \mathbf{x}_i} &= 0.
\end{aligned}$$

To simplify notation, $\psi_i, \mathbf{v}_i, \nabla\psi_i,$ etc. will be used to refer to their function values at $\mathbf{x}_i$ in the subsequent equations of this section. The following formulas are derived in Appendix D:

$$\frac{\partial K}{\partial \dot{R}_i} = 2\pi\rho_\ell (2R_i^3 \dot{R} - 2R_i^2 \psi_i), \tag{22}$$

$$\frac{\partial K}{\partial \mathbf{U}_i} = 2\pi\rho_\ell \left(\frac{1}{3}R_i^3 \mathbf{U}_i - R_i^3 \mathbf{v}_i\right), \tag{23}$$

$$\frac{\partial K}{\partial R_i} = 2\pi\rho_\ell \left(3R_i^2 \dot{R}_i^2 + \frac{1}{2}R_i^2 |\mathbf{U}_i|^2 - 4R_i \dot{R}_i \psi_i - 3R_i^3 \mathbf{U}_i \cdot \mathbf{v}_i + \frac{3}{2}R_i^2 |\mathbf{v}_i|^2 + F\right), \tag{24}$$

$$\frac{\partial K}{\partial \mathbf{x}_i} = 2\pi\rho_\ell (-2R_i^2 \dot{R}_i \mathbf{v}_i + R_i^3 \left(\nabla \mathbf{v}_i\right)^T \cdot (\mathbf{v}_i - \mathbf{U}_i) + G), \tag{25}$$

where F and G involve only $\nabla^2\psi_i, \nabla^3\psi_i, \ldots$, which correspond to spherical harmonics of higher order than a dipole. From the above formulas, we obtain the equations of motion

$$\begin{aligned}\frac{d}{dt}\left(2R_i^3\dot{R}_i - 2R_i^2\psi_i\right) - \Big(3R_i^2\dot{R}_i^2 + \frac{1}{2}R_i^2|\mathbf{U}_i|^2 - 4R_i\dot{R}_i\psi_i - 3R_i^3\mathbf{U}_i\cdot\mathbf{v}_i \\ + \frac{3}{2}R_i^2|\mathbf{v}_i|^2 + F\Big) + 2\frac{p_\infty}{\rho_\ell}\left(-R_0^{3\gamma}R_i^{-3\gamma+2} + R_i^2\right) = 0,\end{aligned} \tag{26}$$

$$\frac{d}{dt}\left(\frac{1}{3}R_i^3\mathbf{U}_i - R_i^3\mathbf{v}_i\right) - \left(-2R_i^2\dot{R}_i\mathbf{v}_i + R_i^3\left(\nabla\mathbf{v}_i\right)^T\cdot(\mathbf{v}_i - \mathbf{U}_i) + G\right) = 0, \tag{27}$$

$$\psi_i = \sum_{i\neq k}\left(\phi_k(\mathbf{x}_i) + I_k\psi_k(\mathbf{x}_i)\right), \tag{28}$$

$$\mathbf{v}_i = \nabla\psi_i. \tag{29}$$

As it stands, the above system is rather intractable for analysis. To proceed further we must make a simplifying approximation, which will be to keep only terms that arise from monopoles and dipoles. Therefore, the F and G terms in (26) and (27) will be ignored. We will also use this approximation to simplify (28) and (29) as follows: from Theorem 3.2 we have

$$I_k\psi_k(\mathbf{x}_i) = -\frac{1}{2}R_k^3\nabla\psi_k(\mathbf{x}_k)\cdot\nabla_{\mathbf{x}_k}\left(\frac{1}{|\mathbf{x}_i - \mathbf{x}_k|}\right) + \text{(higher order harmonics)},$$

which when used with (29) gives

$$I_k\psi_k(\mathbf{x}_i) = -\frac{1}{2}R_k^3\mathbf{v}_k\cdot\nabla_{\mathbf{x}_k}\left(\frac{1}{|\mathbf{x}_i - \mathbf{x}_k|}\right) + \text{(higher order harmonics)}.$$

Next, we combine the above formula with (28) to obtain

$$\begin{aligned}\psi_i = \sum_{k\neq i}\left[-\frac{R_k^2\dot{R}_k}{|\mathbf{x}_i - \mathbf{x}_k|} + \frac{1}{2}R_k^3\nabla_{\mathbf{x}_i}\left(\frac{1}{|\mathbf{x}_i - \mathbf{x}_k|}\right)\cdot(\mathbf{U}_k - \mathbf{v}_k)\right] \\ + \text{(higher order harmonics)}.\end{aligned}$$

Thus by ignoring the terms caused by spherical harmonics of orders higher than dipole, (26)–(29) simplify to

$$\begin{aligned}\frac{d}{dt}(R_i^3\dot{R}_i - R_i^2\psi_i) - \left(\frac{3}{2}R_i^2\dot{R}_i^2 + \frac{1}{4}R_i^2|\mathbf{U}_i|^2 - 2R_i\dot{R}_i\psi_i - \frac{3}{2}R_i^3\mathbf{U}_i\cdot\mathbf{v}_i + \frac{3}{4}R_i^2|\mathbf{v}_i|^2\right) \\ + \frac{p_\infty}{\rho_\ell}(-R_0^{3\gamma}R_i^{-3\gamma+2} + R_i^2) = 0,\end{aligned} \tag{30}$$

$$\frac{d}{dt}\left(\frac{1}{3}R_i^3\mathbf{U}_i - R_i^3\mathbf{v}_i\right) - \left(-2R_i^2\dot{R}_i\mathbf{v}_i + R_i^3\left(\nabla\mathbf{v}_i\right)^T\cdot(\mathbf{v}_i - \mathbf{U}_i)\right) = 0, \tag{31}$$

$$\psi_i = \sum_{k\neq i} -\frac{R_k^2\dot{R}_k}{|\mathbf{x}_i - \mathbf{x}_k|} + \frac{1}{2}R_k^3\nabla_{\mathbf{x}_i}\left(\frac{1}{|\mathbf{x}_i - \mathbf{x}_k|}\right)\cdot(\mathbf{U}_k - \mathbf{v}_k), \tag{32}$$

$$\mathbf{v}_i = \nabla_{\mathbf{x}_i}\psi_i = \sum_{k\neq i}\nabla_{\mathbf{x}_i}\left(-\frac{R_k^2\dot{R}_k}{|\mathbf{x}_i - \mathbf{x}_k|} + \frac{1}{2}R_k^3\nabla_{\mathbf{x}_i}\left(\frac{1}{|\mathbf{x}_i - \mathbf{x}_k|}\right)\cdot(\mathbf{U}_k - \mathbf{v}_k)\right). \tag{33}$$

These equations simplify greatly if we define the following ambient pressure motivated using Bernoulli's law. The ambient pressure associated with the ith bubble is defined as

$$\frac{p_i}{\rho_\ell} = \frac{p_\infty}{\rho_l} - \frac{\partial \psi_i}{\partial t} - \frac{1}{2}|\mathbf{v}_i|^2. \tag{34}$$

Then (30) and (31) can be written as

$$R_i \ddot{R}_i + \frac{3}{2}\dot{R}_i^2 - \frac{1}{4}|\mathbf{U}_i - \mathbf{v}_i|^2 + \frac{1}{\rho_\ell}(p_i - p_{g,i}) = 0, \tag{35}$$

$$\frac{1}{3}\dot{\mathbf{U}}_i - \frac{D\mathbf{v}_i}{Dt} + \frac{\dot{R}_i}{R_i}(\mathbf{U}_i - \mathbf{v}_i) = (\mathbf{v}_i - \mathbf{U}_i) \times (\nabla \times \mathbf{v}_i), \tag{36}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_i \cdot \nabla \quad \text{and} \quad p_{g,i} = p_\infty \left(\frac{R_0}{R_i}\right)^{3\gamma}.$$

In arriving at (36) we have used the identity

$$-(\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \mathbf{v})^T \cdot \mathbf{u} = \mathbf{u} \times (\nabla \times \mathbf{v}).$$

In deriving these equations of motion we have not assumed that $\nabla \times \mathbf{v}_i = 0$; however, it is apparent from (33) that $\nabla \times \mathbf{v}_i = 0$ since $\mathbf{v}_i = \nabla \psi_i$. This means the right-hand side of (36) is zero. Nevertheless, we shall leave (36) in the form we have written since, as we shall see later, the continuum limit of $\mathbf{v}_i$ is not necessarily curl free.

3.3.1. Bubble motion in nonuniform flows. The motion of a bubble in a nonuniform potential flow has been the subject of much interest. If we consider a massless rigid spherical bubble with velocity $\mathbf{U}$ moving in liquid with an ambient velocity $\mathbf{v}$, then the equation of motion is

$$\frac{1}{2}\dot{U} = \frac{3}{2}\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right), \tag{37}$$

where $\mathbf{v}$ is derived from a potential flow, thus $\nabla \times \mathbf{v} = 0$. This equation has been derived by Voinov, Voinov, and Petrov [54], Landweber and Miloh [27], van Wijngaarden [51], and Galper and Miloh [14]. Galper and Miloh [14, 15] also derive extensions of this formula for more complex problems. If $\dot{R}_i = 0$, then we see that (36) reduces to (37), where we have used $\nabla \times \mathbf{v}_i = 0$.

Therefore we see that our computation is in agreement with well-known results. The main contribution of our work is in finding a self-consistent expression for the ambient liquid velocity produced by the motion of the other bubbles.

4. Continuum limit. In the previous section, we derived the equations of motion for a finite number of bubbles. In this section, we will take the continuum limit to obtain our effective equations. This approach is similar to that used by solid state physics to obtain effective equations; see, for example, Batteh and Powell [5] or Rosenau [35]. It also very close to the approach used by Caflisch et al. [9]. This approach is expected to give a faithful approximation, provided that the wavelength of interest is considerably longer than the distance between particles. One of the important parts of this section is taking the continuum limit of (32). This is obtained by approximating the summation by an integration. Let us explain this with the following example.

4.0.1. Example. Consider the situation with N point charges located at $\mathbf{x}_j$ with charge q_j, where $j = 1, \ldots, N$. The ambient electric potential at $\mathbf{x}_i$ is

$$\psi_i(\mathbf{x}_i) = \sum_{i \neq j}^{N} \frac{q_j}{|\mathbf{x}_i - \mathbf{x}_j|}. \tag{38}$$

We suppose that there exists a smooth function $q(\mathbf{x})$ such that $q_j = q(\mathbf{x}_j)$; then this summation can be approximated as

$$\psi(\mathbf{x}) = \int \frac{\rho(\mathbf{y})q(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}, \tag{39}$$

where $\rho(\mathbf{x})$ is the number of particles per unit volume. Since $-(4\pi|\mathbf{x}|)^{-1}$ is the fundamental solution of Laplace's equation in three space dimensions, then it follows that (39) is equivalent to

$$\triangle\psi = -4\pi q(\mathbf{x})\rho(\mathbf{x}). \tag{40}$$

4.1. Effective equations for bubbly flows. Our equations of motion for a finite number of bubbles are given by (35) and (36), with the ambient field determined from (32), (33), and (34). When passing to the continuum limit, we first assume that there exist functions $R(\mathbf{x}, t)$ and $\mathbf{U}(\mathbf{x}, t)$ such that $R_k(t) = R(\mathbf{x}_k, t)$ and $\mathbf{U}_k(t) = \mathbf{U}(\mathbf{x}_k, t)$. We note that this assumption indicates that we assume that, on length scales less than the wavelength of interest, the bubbles are all "doing the same thing." This assumption implies that neighboring bubbles are oscillating coherently and moving with the same velocity. If we assume that nearby bubbles have the same velocity, then this means that we are studying "cold" bubbly flows; in other words, we are ignoring effects of the fluctuation of the bubbles' velocity. These effects have been studied in simpler models of bubbly flows by Russo and Smereka [37] and Herrero, Lucquin-Desreux, and Perthame [20]. The effects of incoherent bubble oscillations have been considered by, for example, Carstensen and Foldy [11] and Smereka [44] for models that ignore the effects of bubble translation. We also point out that this same assumption was used by Zhang and Prosperetti [57] in what they call sharply peaked probability distributions. As pointed out in [44], this same assumption was used by van Wijngaarden [48] and Caflisch et al. [10].

Now we take the continuum limit of (32), following the approach outlined in the example above, and we find

$$\psi(\mathbf{x}, t) = \int \rho \left(-\frac{R^2(\mathbf{y}, t)\dot{R}(\mathbf{y}, t)}{|\mathbf{x} - \mathbf{y}|} + \frac{1}{2} R^3(\mathbf{y}, t) \nabla_{\mathbf{x}} \left(\frac{1}{|\mathbf{x} - \mathbf{y}|} \right) \cdot (\mathbf{U}(\mathbf{y}, t) - \mathbf{v}(\mathbf{y}, t)) \right) d\mathbf{y},$$

where $\dot{}$ denotes $\frac{d}{dt} = \partial_t + \mathbf{U} \cdot \nabla$ and $\rho = \rho(\mathbf{y}, t)$ is the number of bubbles per unit volume. ρ satisfies the following conservation equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \tag{41}$$

and is a statement of particle conservation.

Using integration by parts on our expression for ψ, we have

$$\psi(\mathbf{x}, t) = \int \left(-\frac{\rho R^2 \dot{R}}{|\mathbf{x} - \mathbf{y}|} + \frac{1}{2} \frac{\nabla \cdot \left(\rho R^3 (\mathbf{U} - \mathbf{v})\right)}{|\mathbf{x} - \mathbf{y}|} \right) d\mathbf{y},$$

which is equivalent to

$$\triangle\psi = -4\pi\left(-\rho R^2\dot{R} + \frac{1}{2}\nabla\cdot\left(\rho R^3(\mathbf{U}-\mathbf{v})\right)\right).$$

If we use the void fraction $\beta = \frac{4}{3}\pi R^3\rho$ instead of ρ, the above equation becomes

$$\triangle\psi = \frac{3\beta}{R}\dot{R} - \frac{3}{2}\nabla\left(\cdot\beta(\mathbf{U}-\mathbf{v})\right). \tag{42}$$

In a similar way we see that the continuum limit of (33) is

$$\mathbf{v} = \int\left(-R^2\dot{R}\nabla_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) + \frac{1}{2}R^3\nabla^2_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\cdot(\mathbf{U}-\mathbf{v})d\mathbf{y}\right). \tag{43}$$

Caution has to be taken in evaluating the second part of the above integral, as

$$\int\nabla^2_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\cdot\mathbf{p}d\mathbf{y}$$

is singular and the integrand is not integrable. We take the principal value, which is defined. This is justified because bubble i is not in the original sum of (33). From Smereka [43, (27)] we have

$$⨍\nabla^2_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\cdot\mathbf{p}(\mathbf{y})d\mathbf{y} = \frac{4\pi}{3}\mathbf{p}(x) + \nabla_{\mathbf{x}}\int_V\nabla_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\cdot\mathbf{p}(\mathbf{y})d\mathbf{y}, \tag{44}$$

where $⨍$ is the principal value integral. Using the above formula in (43), we obtain

$$\mathbf{v} = \nabla\psi + \frac{\beta}{2}(\mathbf{U}-\mathbf{v}). \tag{45}$$

To obtain an expression for β we start with (41) and use $\beta = \frac{4}{3}\pi R^3\rho$ to obtain

$$\frac{d}{dt}\left(\frac{3\beta}{4\pi R^3}\right) + \left(\frac{3\beta}{4\pi R^3}\right)\nabla\cdot U = 0,$$

which can be simplified to

$$\frac{d\beta}{dt} - \frac{3\beta}{R}\frac{dR}{dt} + \beta\nabla\cdot\mathbf{U} = 0. \tag{46}$$

This is the conservation of volume for the gas phase. If we take the gradient of (45) and substitute it into (42), we find

$$\nabla\cdot(\beta\mathbf{U} + (1-\beta)\mathbf{v}) - \frac{3\beta}{R}\frac{dR}{dt} = 0. \tag{47}$$

This is exactly the conservation of total volume. This indicates that, to the level of our approximation, $\mathbf{v}$ is also the volume averaged liquid velocity. We can make this point even more transparent by subtracting (46) from (47), to obtain

$$\frac{\partial(1-\beta)}{\partial t} + \nabla\cdot((1-\beta)\mathbf{v}) = 0. \tag{48}$$

This is a statement of conservation of liquid volume.

The continuum limit of (35) and (36) are obtained by realizing that $\frac{d}{dt}$ is the material derivative. Collecting our results, we find the following set of effective equations:

$$R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2 - \frac{1}{4}|\mathbf{U}-\mathbf{v}|^2 + \frac{p-p_g}{\rho_\ell} = 0, \tag{49}$$

$$\frac{1}{3}\frac{d\mathbf{U}}{dt} - \frac{D\mathbf{v}}{Dt} + \frac{1}{R}\frac{dR}{dt}(\mathbf{U}-\mathbf{v}) + (\mathbf{U}-\mathbf{v}) \times (\nabla \times \mathbf{v}) = 0, \tag{50}$$

$$\triangle\psi - \frac{3\beta}{R}\frac{dR}{dt} + \frac{3}{2}\nabla\cdot(\beta(\mathbf{U}-\mathbf{v})) = 0, \tag{51}$$

$$\mathbf{v} - \nabla\psi - \frac{\beta}{2}(\mathbf{U}-\mathbf{v}) = 0, \tag{52}$$

$$\frac{1}{2}\mathbf{v}^2 + \frac{\partial\psi}{\partial t} + \frac{p-p_\infty}{\rho_\ell} = 0, \tag{53}$$

$$\frac{\partial\beta}{\partial t} + \nabla\cdot(\beta\mathbf{U}) - \frac{3\beta}{R}\frac{dR}{dt} = 0, \tag{54}$$

$$p_g - p_\infty\left(\frac{R_0}{R}\right)^{3\gamma} = 0, \tag{55}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla \quad \text{and} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla.$$

To summarize, in the above set of effective equations, (49) and (50) are the continuum versions of (35) and (36). Equations (51), (52), and (53) are the continuum versions of (32), (33), and (34), respectively. Finally, (54) is a statement of bubble number conservation (equivalently, conservation of liquid volume), and (55) is the equation of state for the gas contained in the bubbles. The equation for the conservation of liquid volume, (48), shows that the ambient velocity is well approximated by the average liquid velocity.

We observe from (52) that

$$\nabla\times\mathbf{v} = \nabla\psi\times\nabla\left(\frac{2}{2+\beta}\right) + \nabla\times(\beta\mathbf{U}),$$

which is not necessarily zero. This may seem strange since it appears from (33) that $\mathbf{v}_i$ is curl free. However, it is important to note that $\mathbf{v}_i(\mathbf{x})$ has singularities when $\mathbf{x} = \mathbf{x}_k$. In the discrete case these singularities are not important, as is clear from (33). However, when we pass to the continuum limit, these singularities become source terms which cause the continuum limit to have a nonzero curl.

A simple example of this behavior can be understood using the example given in subsection 4.01. It is clear that (38) is a harmonic function of $\mathbf{x}_i$ ($\triangle_{\mathbf{x}_i}\psi_i = 0$) except when $\mathbf{x}_i = \mathbf{x}_k$. The expression for (38) is singular at $\mathbf{x}_i = \mathbf{x}_k$, corresponding to the location of the charges. It is evident from (40) that the continuum limit, (39), of (38) is not harmonic due the presence of the charges.

In a similar way, it follows that while the discrete ambient velocity field $\mathbf{v}_i$ is curl free, the same is not true of the continuum limit. This behavior is not uncommon; for example, in the theory of dielectrics the local electric field is curl free, whereas the ambient field of a continuum of dipoles is not curl free (see, for example Lorrain and Corson [29]). We are not claiming that this flow has vorticity. The ambient liquid velocity has a nonzero curl, whereas the local liquid velocity is curl free. This may sound contradictory, but it is not; the ambient liquid velocity does not satisfy the Euler equation. Another way to look at this term is that it reflects the vorticity present on the surface of each bubble, and the ambient liquid velocity is a homogenized liquid velocity that accounts for the bubbles. In fact, the net vorticity produced by a single bubble is zero. Therefore if we had a homogeneous suspension of bubbles all moving with same velocity, then $\mathbf{v}$ would be constant in space and $\nabla \times \mathbf{v} = 0$.

It is interesting to compare (50) with work of Auton, Hunt, and Prud'homme [1]. In this work the authors compute the force on a bubble in a nonuniform flow with vorticity. From their work it follows that the equation of motion of the bubble (in our notation) is given by

$$C_M \left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) = (1 + C_M) \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) - C_L (\mathbf{U} - \mathbf{v}) \times \boldsymbol{\omega}, \tag{56}$$

where C_M is the added mass coefficient, C_L is the rotational lift coefficient, and $\boldsymbol{\omega}$ is the liquid vorticity. If we choose $C_M = \frac{1}{2}$ and $C_L = \frac{3}{2}$, then we see that our result is the same as theirs (using $\dot{R} = 0$). A priori we should not expect any close relation between (56) and (50), since in (50), $\nabla \times \mathbf{v}$ is not the vorticity of the liquid. In fact, Auton, Hunt, and Prud'homme argue for a spherical bubble that $C_L = \frac{1}{2}$, whereas we find $C_L = \frac{3}{2}$.

4.2. Sound speed. We now study sound propagation for one dimensional flows, and therefore we assume that our dependent variables depend on only one space coordinate, which we take to be x. We linearize (49) through (55) around the equilibrium

$$R = R_0, \quad U = v = \psi = 0, \quad \beta = \beta_0,$$

and obtain the linearized equations

$$\begin{aligned}
&\frac{\partial^2 R}{\partial t^2} + \omega_0^2 R - \frac{1}{R_0} \frac{\partial \psi}{\partial t} = 0, \\
&\frac{\partial U}{\partial t} - 3 \frac{\partial v}{\partial t} = 0, \\
&\frac{\partial^2 \psi}{\partial x^2} = \frac{3\beta_0}{R_0} \frac{\partial R}{\partial t} - \frac{3}{2} \beta_0 \frac{\partial (U - v)}{\partial x}, \\
&v = \frac{\partial \psi}{\partial x} + \frac{\beta_0}{2} (U - v), \\
&\frac{\partial \beta}{\partial t} - \frac{3\beta_0}{R_0} \frac{\partial R}{\partial t} + \beta_0 \frac{\partial U}{\partial x} = 0,
\end{aligned}$$

where the unsubscripted variables represent perturbations from equilibrium and

$$\omega_0 = \sqrt{\frac{3\gamma P_\infty}{\rho_\ell R_0^2}}$$

is the natural frequency of a single bubble in an unbounded fluid.

We let $(R, U, v, \psi, \beta) = (A, B, C, D, E)e^{i(\omega t - kx)}$ and find

$$\begin{pmatrix} -\omega^2 + \omega_0^2 & 0 & 0 & -\frac{i\omega}{R_0} & 0 \\ 0 & i\omega & -3i\omega & 0 & 0 \\ -\frac{3i\omega\beta_0}{R_0} & -\frac{3ik\beta_0}{2} & \frac{3ik\beta_0}{2} & -k^2 & 0 \\ 0 & -\frac{\beta_0}{2} & 1 + \frac{\beta_0}{2} & ik & 0 \\ -\frac{3\beta_0 i\omega}{R_0} & -ik\beta_0 & 0 & 0 & i\omega \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \end{pmatrix} = 0.$$

The above equations will have nontrivial solutions when the determinant is zero. Thus, we obtain the dispersion relation

$$3\omega^2\beta_0(1-\beta_0) = k^2 R_0^2(1+2\beta_0)(\omega_0^2 - \omega^2).$$

The effective sound speed is $c = \frac{\omega}{k}$ and

$$c^2 = \frac{R_0^2(1+2\beta_0)(\omega_0^2 - \omega^2)}{3\beta_0(1-\beta_0)}. \tag{57}$$

If we let $\omega \to 0$, we have

$$c_0^2 = \frac{R_0^2(1+2\beta_0)\omega_0^2}{3\beta_0(1-\beta_0)},$$

which is the same as the expression given by Crespo [12] and Caflisch et al. [10]. The sound speed in (57) is also in agreement with Sangani [38]. We note that we have assumed that the liquid is incompressible; therefore to compare with other investigations one must consider the case $C_\ell \to \infty$, where C_ℓ is the speed of sound in the liquid region.

4.3. Void waves. Void waves have been observed, and various properties such as wave speed have been measured. Typically void waves travel at speeds much slower than sound waves. This means that void waves and sound waves interact weakly, and we will not make any significant error if we assume the bubble radius is fixed. In this case our system of equations becomes

$$\frac{1}{3}\frac{d\mathbf{U}}{dt} - \frac{D\mathbf{v}}{Dt} + (\mathbf{U} - \mathbf{v}) \times (\nabla \times \mathbf{v}) = 0, \tag{58}$$

$$\triangle\psi + \frac{3}{2}\nabla \cdot (\beta(\mathbf{U} - \mathbf{v})) = 0, \tag{59}$$

$$\mathbf{v} - \nabla\psi - \frac{\beta}{2}(\mathbf{U} - \mathbf{v}) = 0, \tag{60}$$

$$\frac{\partial\beta}{\partial t} + \nabla \cdot (\beta\mathbf{U}) = 0. \tag{61}$$

These equations simplify greatly if we consider flows in one spatial dimension. In this situation (59) and (60) become

$$\frac{\partial^2\psi}{\partial x^2} + \frac{3}{2}\frac{\partial}{\partial x}(\beta(U - v)) = 0 \quad \text{and} \quad v - \frac{\partial\psi}{\partial x} - \frac{\beta}{2}(U - v) = 0.$$

We eliminate ψ to obtain

$$\frac{\partial}{\partial x}\left((1-\beta)v + \beta U\right) = 0.$$

Since we are in the situation where the volume flux is zero, the above equation indicates that

$$v = \frac{-\beta U}{(1-\beta)}. \tag{62}$$

In the one dimensional case, (58) and (61) become

$$\frac{1}{3}\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}\right) - \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = 0, \tag{63}$$

$$\frac{\partial \beta}{\partial t} + \frac{\partial}{\partial x}(\beta U) = 0. \tag{64}$$

If we multiply (63) by 3/2 and use (62), we find

$$\frac{\partial}{\partial t}\left(\frac{(1+2\beta)}{2(1-\beta)}U\right) + \frac{\partial}{\partial x}\left(\frac{(1-2\beta-2\beta^2)}{4(1-\beta)^2}U^2\right) = 0. \tag{65}$$

Next we consider a new variable

$$M = \frac{\beta U}{h(\beta)}, \quad \text{where} \quad h(\beta) = \frac{2\beta(1-\beta)}{1+2\beta}; \tag{66}$$

then (64) and (65) can be written as the following system:

$$\begin{aligned} &\frac{\partial \beta}{\partial t} + \frac{\partial}{\partial x}(h(\beta)M) = 0, \\ &\frac{\partial M}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}h'(\beta)M^2\right) = 0. \end{aligned} \tag{67}$$

This set of conservation laws can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} \beta \\ M \end{pmatrix} + \mathsf{A}\frac{\partial}{\partial x}\begin{pmatrix} \beta \\ M \end{pmatrix} = 0, \quad \text{where} \quad \mathsf{A} = \begin{pmatrix} h'(\beta)M & h(\beta) \\ \frac{1}{2}h''(\beta)M^2 & h'(\beta)M \end{pmatrix}. \tag{68}$$

The eigenvalues of A are

$$\lambda = M\left(h'(\beta) \pm \sqrt{\frac{1}{2}h(\beta)h''(\beta)}\right).$$

Upon substituting the expression for $h(\beta)$, we find

$$\lambda = \frac{2M}{(1+2\beta)^2}\left(1-2\beta-2\beta^2 \pm i\sqrt{3\beta(1-\beta)}\right). \tag{69}$$

This dispersion relation shows that the Fourier modes will increase in magnitude at a rate proportional to wave number. This indicates that a spatially uniform bubbly flow is unstable to all perturbations. In fact, the initial value problem for (68) is ill-posed. This is consistent with the bubble clustering observed in the numerical simulations done by Sangani and Didwania [39] and Smereka [42]. In the next section, we will compute the dispersion relation when viscosity and gravity are considered. We will also offer an explanation of the discrepancy between experiments and numerical simulations.

4.4. Comparison with previous work. Geurst [17, 18] derived a set of equations for two-phase flow using a variational method. The same set of equations were derived by Wallis [47] and Pauchon and Smereka [34] using different approaches. The equations contained one phenomenological relation, denoted $m_G(\beta)$, which Wallis calls the exteria. Pauchon and Smereka showed that Geurst's equations simplify greatly in the frame of reference where the volume flux is zero. If we use $m_G(\beta) = \beta/2$ in Guerst's equations (as written by Pauchon and Smereka), we find that they are identical to (67). (Note, however, that Pauchon and Smereka use $\Gamma(\beta)$, which is $1/h(\beta)$). It should also be noted that from the work of Smereka and Milton [41] one can show that the exteria, $m_G(\beta)$, is related to the virtual mass in the zero volume flux frame $m(\beta)$, as follows:

$$m_G(\beta) = \frac{m(\beta)}{\rho_\ell}(1-\beta) - \beta^2. \tag{70}$$

The virtual mass in the zero volume flux frame has been calculated by Zuber [59], van Wijngaarden [50], Biesheuvel and Spolestra [7], Wallis [47], and Smereka and Milton [41]. Zuber's result was

$$m(\beta) = \rho_\ell \frac{\beta}{2}\left(\frac{1+2\beta}{1-\beta}\right).$$

The results of the other investigators were similar. Smereka and Milton showed that Zuber's result was exact for a certain type of bubbly flow. If we substitute Zuber's result into (70), then we find $m_G(\beta) = \beta/2$. Thus we conclude that (67) is the same as that derived by Geurst when Zuber's expression for the virtual mass is used.

5. Effects of liquid viscosity and gravity. In this section, we consider the effects of liquid viscosity and gravity in an effort to understand the dynamics of void waves. We shall assume that the bubble radii are unchanging and of identical sizes.

The effective equations are derived, and the void wave speed obtained is in good agreement with experimental data. We also offer an explanation of why bubble clustering is not observed in experiments.

5.1. Equations of motion. We shall proceed in a fashion similar to that in section 3. The Lagrangian is given by

$$\mathcal{L} = \mathcal{K} - \mathcal{U},$$

where $\mathcal{K}$ is given by (20). The potential energy is modified by gravity as follows:

$$\mathcal{U} = \mathcal{U}_g + \rho_\ell g \sum_{k=1}^{N} \frac{4}{3}\pi R_k^3 z_k,$$

where z_k is the z coordinate of the kth bubble and $\mathcal{U}_g$ is given by (21). We shall include effects of liquid viscosity by using a dissipation function denoted as $\mathcal{D}$. The equations of motion are now

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \mathbf{U}_i} - \frac{\partial \mathcal{L}}{\partial \mathbf{x}_i} = \frac{1}{2}\frac{\partial \mathcal{D}}{\partial \mathbf{U}_i}. \tag{71}$$

The amount of energy dissipated is given by

$$\mathcal{D} = \int_{V_\ell} D d\mathbf{x}, \qquad \text{where} \quad D = 2\mu\varepsilon_{ij}\cdot\varepsilon_{ij},$$

where ε_{ij} is the rate-of-strain tensor.

We shall assume that the Reynolds number is high, so the flow is close to potential flow except in the thin boundary layer wrapped around each bubble. We shall further assume that no significant amount of energy dissipates in the boundary layer. The verification of this assumption for one bubble can be found in Moore [31]. With this assumption one finds that

$$\varepsilon_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}.$$

We can check that

$$D = \mu \Delta E, \qquad \text{where} \quad E = \nabla\phi \cdot \nabla\phi.$$

Using Green's theorem, we have

$$\mathcal{D} = \int_{V_\ell} \mu \Delta E dv = -\mu \int_S \nabla E \cdot \mathbf{n} ds = -\mu \int_S \frac{\partial E}{\partial n} ds = -\mu \int_S \frac{\partial(\nabla\phi \cdot \nabla\phi)}{\partial n} ds, \tag{72}$$

where the integral is taken on the surface of the bubbles, and $\mathbf{n}$ is an outward normal vector.

5.2. Calculation of drag. In this section we calculate the drag force of a bubble in the presence of a finite number of bubbles. We begin with the case of a single bubble.

5.2.1. Single bubble. We consider a single bubble, moving with a fixed radius and a translational velocity $\mathbf{U}$, in a fluid with a constant ambient velocity $\mathbf{v}_\infty$. The velocity potential in this case is

$$\phi = \frac{1}{2} R^3 \nabla_r \left(\frac{1}{|\mathbf{r} - \mathbf{x}|} \right) \cdot (\mathbf{U} - \mathbf{v}_\infty) + \mathbf{v}_\infty \cdot \mathbf{r}.$$

The energy dissipation is computed using (72) and found to be

$$\mathcal{D} = 12\pi\mu R |\mathbf{U} - \mathbf{v}_\infty|^2. \tag{73}$$

The drag on the bubble is then given by

$$\mathbf{F} = \frac{1}{2} \frac{\partial \mathcal{D}}{\partial \mathbf{U}} = 12\pi\mu R(\mathbf{U} - \mathbf{v}_\infty). \tag{74}$$

Levich [28] derived (74) using the method outlined here. Moore [31] determined the drag force by computing the pressure distribution around the bubble. Kang and Leal [23] and Stone [46] provide alternate derivations.

5.2.2. Two bubbles. For the case of two bubbles in an infinite liquid, van Wijngaarden and Kapteyn [52], using the energy dissipation argument, derived

$$\mathbf{F} = -12\pi\mu R(\mathbf{U} - 2\mathbf{v}_{ind}), \tag{75}$$

where $\mathbf{v}_{ind}$ is the velocity generated by the other bubble.

5.2.3. N bubbles. With the expression of the velocity potential (14), we calculate $\mathcal{D}$ in Appendix E. After neglecting terms caused by spherical harmonics of orders higher than dipole, we have

$$\mathcal{D} = 12\pi\mu R \sum_{i=1}^{N} |\mathbf{U}_i - \mathbf{v}_i(\mathbf{x}_i)|^2. \tag{76}$$

From this equation we see that the dissipation due to the ith bubble is

$$12\pi\mu R|\mathbf{U}_i - \mathbf{v}_i(\mathbf{x}_i)|^2.$$

This compares closely with (73), with $\mathbf{v}_\infty$ replaced by the ambient field of the ith bubble ($\mathbf{v}_i$). Nevertheless, the drag force will be different from (74) since the ambient velocity of the ith bubble will depend on the velocities of all of the bubbles. In Appendix E we compute the drag force and find

$$\frac{1}{2}\frac{\partial \mathcal{D}}{\partial \mathbf{U}_i} = 12\pi\mu R(\mathbf{U}_i - 2\mathbf{v}_i(\mathbf{x}_i) - \mathbf{w}_i(\mathbf{x}_i)), \tag{77}$$

where

$$\mathbf{w}_i(\mathbf{r}) = -\sum_{j\neq i}\frac{1}{2}R^3\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{x}_j|}\right)\cdot\mathbf{v}_j.$$

This compares closely with the result by van Wijngaarden and Kapteyn [52].

The continuum limit of (77) is

$$12\pi\mu R(\mathbf{U} - 2\mathbf{v} - \mathbf{w}),$$

where $\mathbf{v}$ is given by (59) and (60); $\mathbf{w}$ is determined from

$$\triangle\chi - \frac{3}{2}\nabla\cdot(\beta\mathbf{v}) = 0, \tag{78}$$

$$\mathbf{w} - \nabla\chi + \frac{\beta\mathbf{v}}{2} = 0. \tag{79}$$

In one space dimension, (78) and (79) simplify to

$$\frac{\partial^2\chi}{\partial x^2} = \frac{3}{2}\frac{\partial\beta v}{\partial x} \quad \text{and} \quad w = \frac{\partial\chi}{\partial x} - \frac{1}{2}\beta v,$$

from which it follows that

$$\frac{\partial w}{\partial x} - \frac{\partial(\beta v)}{\partial x} = 0.$$

Since w must vanish if v vanishes, then we find

$$w = \beta v.$$

Therefore the drag force is

$$12\pi\mu R(U - (2+\beta)v). \tag{80}$$

Using the expression for v given by (62), we find that the drag force is

$$12\pi\mu R U\frac{1+\beta+\beta^2}{1-\beta}. \tag{81}$$

We can write this formula in a different form by noticing that the bubble's velocity relative to the ambient liquid velocity is

$$U - v = \frac{U}{1-\beta}.$$

Therefore we can rewrite the drag force as

$$12\pi\mu R(1+\beta+\beta^2)(U-v).$$

We have shown, by following a procedure similar to that outlined in Appendix E, that the formula above is valid for any value of the volume flux.

5.3. Void waves. With the drag force computed, we can now modify our model for void waves to include gravity and liquid viscosity. Following the approach previously outlined, we find that (71) becomes, in the continuum limit,

$$\frac{1}{3}\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}\right)-\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)=\frac{2}{3}g\left(1-\frac{U}{U_\infty}\frac{1+\beta+\beta^2}{1-\beta}\right), \tag{82}$$

$$\frac{\partial \beta}{\partial t}+\frac{\partial(\beta U)}{\partial x}=0, \tag{83}$$

where v is given by (62) and

$$U_\infty=\frac{R^2\rho_\ell g}{9\mu}$$

is the steady speed of a single bubble rising in an infinite fluid under the force of gravity. Next we multiply (82) by $3/2$, use the expression for v, and rewrite (82) as

$$\frac{\partial}{\partial t}\left(\frac{(1+2\beta)}{2(1-\beta)}U\right)+\frac{\partial}{\partial x}\left(\frac{(1-2\beta-2\beta^2)}{4(1-\beta)^2}U^2\right)=g-\frac{gU}{U_\infty}\left(\frac{(1+\beta+\beta^2)}{1-\beta}\right). \tag{84}$$

Our model for void wave propagation including dissipation is then given by (83) and (84).

It is easy to verify that (83) and (84) have the equilibrium solution $\beta=\beta_0$ and $U=U_0$, where

$$U_0=\frac{1-\beta_0}{1+\beta_0+\beta_0^2}U_\infty. \tag{85}$$

This corresponds to a spatially uniform mixture of bubbles rising due to gravity in the zero volume flux frame of reference. The rise speed of the bubbles is given by (85). The prediction of (85) is in good agreement with the experimental data reported by Lammers and Biesheuvel [26] as shown in Figure 1 below. This result seems somewhat paradoxical; it has been shown by Sangani and Didwania [39], Smereka [42], and van Wijngaarden [53] that, in the context of potential theory, there is not a stable steady homogeneous distribution of rising bubbles; the key word here is stable. We shall now show that this steady solution that we have calculated is in fact unstable, in agreement with [39, 42, 53]. In fact we conjecture that this steady state is only weakly unstable, which is why (85) is in good agreement with experimental data.

Next we wish to examine the stability of this equilibrium solution. For this purpose it is useful to use the change of variables described in section 3. That is, we consider $M=\beta U/h(\beta)$; then (83) and (84) can be written as the following system:

$$\begin{aligned}&\frac{\partial \beta}{\partial t}+\frac{\partial}{\partial x}\left(h(\beta)M\right)=0,\\ &\frac{\partial M}{\partial t}+\frac{\partial}{\partial x}\left(\frac{1}{2}h'(\beta)M^2\right)=g\left(1-\frac{M}{M_0(\beta)}\right),\end{aligned} \tag{86}$$

where $h(\beta)$ is given by (66) and

$$M_0(\beta)=U_\infty\frac{1+2\beta}{2(1+\beta+\beta^2)}.$$

In these variables the equilibrium solution is $(\beta, M) = (\beta_0, M_0(\beta_0))$. If we linearize (86) about the equilibrium solution, we obtain the following linear system, with M and β now being the linearized variables:

$$\frac{\partial}{\partial t}\begin{pmatrix} \beta \\ M \end{pmatrix} + \mathsf{A}_0 \frac{\partial}{\partial x}\begin{pmatrix} \beta \\ M \end{pmatrix} = \mathsf{R}\begin{pmatrix} \beta \\ M \end{pmatrix}, \tag{87}$$

where

$$\mathsf{A}_0 = \begin{pmatrix} h'(\beta_0)M_0 & h(\beta_0) \\ \frac{1}{2}h''(\beta_0)M_0^2 & h'(\beta_0)M_0 \end{pmatrix} \quad \text{and} \quad \mathsf{R} = \frac{g}{M_0(\beta_0)}\begin{pmatrix} 0 & 0 \\ M_0'(\beta_0) & -1 \end{pmatrix}.$$

Next we look for solutions of the form $(\beta, M) = (A, B)e^{i(\omega t - kx)}$. We find that there will be solutions of this form, provided that the following dispersion relationship is satisfied:

$$z^2 + \frac{g}{ikM_0(\beta_0)}z - h(\beta_0)\left(\frac{1}{2}h''(\beta_0)M_0^2(\beta_0) + \frac{gM_0'(\beta_0)}{ikM_0(\beta_0)}\right) = 0, \tag{88}$$

with $z = c - h'(\beta_0)M_0(\beta_0)$, where $c = \frac{\omega}{k}$ is the phase speed. Solving the above equation for z reveals that z always has complex roots, indicating that the initial value problem for (87) is ill-posed. Hence, the growth rate of a Fourier mode is proporational to its wave number. This seems consistent with numerical simulations of bubble clustering. In these simulations a spatially uniform distribution of bubbles quickly assembles into horizontal clusters of bubbles (see Sangani and Didwania [39] and Smereka [42]). However, in experiments such behavior is not observed and is inconsistent with experimental observations of void wave propagation. Nevertheless, we shall see below that our model can predict some phenomena seen in experiments. To this end, let us then consider situations where the wavelengths tend to be large and therefore k is small. For small k the dispersion relationship has the two solutions

$$c_1 = \frac{ig}{M_0(\beta)k} + h'(\beta_0)M_0(\beta_0) - h(\beta_0)M_0'(\beta_0) + ic_I + O(k^2),$$

$$c_2 = (h(\beta_0)M_0(\beta_0))' - ic_I + O(k^2),$$

where

$$c_I = \frac{kh(\beta_0)M_0(\beta_0)}{2g}\left(2h(\beta_0)(M_0'(\beta_0))^2 - M_0^2(\beta_0)h''(\beta_0)\right).$$

Substituting in our expressions for h and M_0, these become

$$c_1 = i\frac{2g}{kU_\infty}\frac{1+\beta_0+\beta_0^2}{1+2\beta_0} + U_\infty(1 - 6\beta_0 + O(\beta_0^2)) + ic_I,$$

$$c_2 = c_R - ic_I,$$

where

$$c_R = \frac{1 - 2\beta_0 - 2\beta_0^2}{(1+\beta_0+\beta_0^2)^2}U_\infty \tag{89}$$

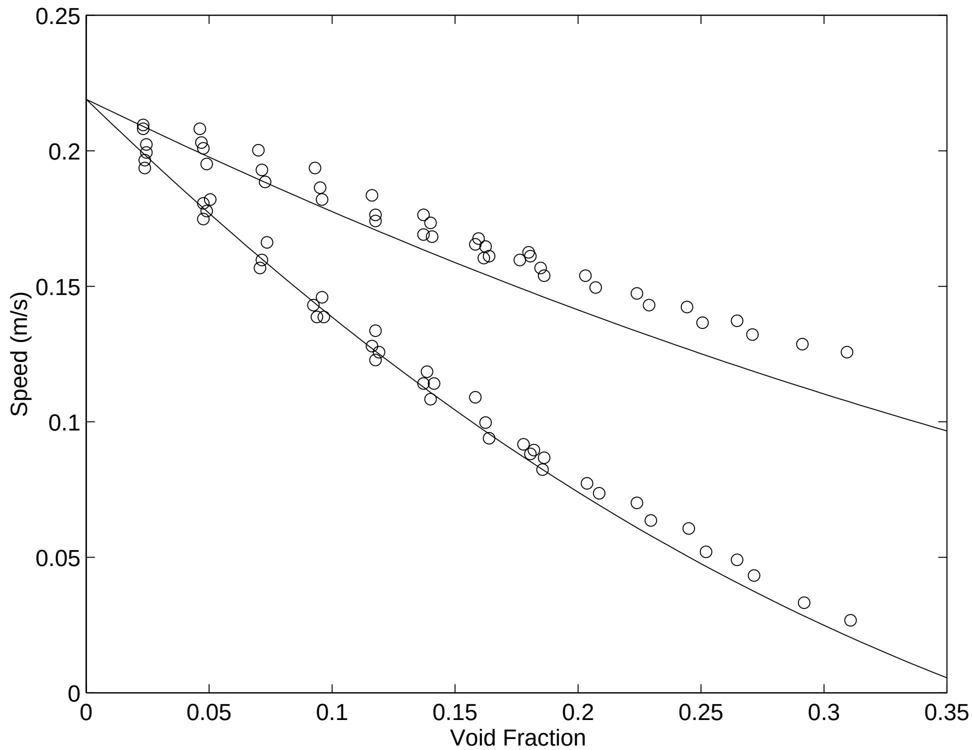

FIG. 1. *The upper curve shows a plot of the predicted bubble rise speed using* (85), *and the lower curve shows the predicted void wave speed using* (89). *We have used* $U_\infty = .219$ *m/sec. The circles are the experimental findings of Lammers and Biesheuvel* [26], *as is the value for* U_∞.

and

$$c_I = \frac{U_\infty^3 k}{2g}(3\beta_0 + O(\beta_0^2)). \tag{90}$$

In the expression for c_1 we observe that its imaginary part is positive. This indicates that this mode decays. This corresponds to the relaxation of the bubble's speed to the equilibrium speed. The second mode corresponds to void waves. It predicts that the void waves will move with speed c_R and will grow with a rate given by $c_I k$. Figure 1 shows a plot of c_R as function of the volume fraction along with experimental data. The agreement is good.

In the experiments of Lammers and Biesheuvel [26] the rise speed of a single bubble was approximately 25 cm/sec, the void fraction was in the range 0 to 0.4, and the frequency of naturally occurring void waves was approximately 1 Hz (see Biesheuvel and Gorissen [6]). Using our expression for the real part of the wave speed, we can estimate that this corresponds to disturbances with a wavelength of approximately 27 cm ($k \approx .2$). The growth rate of these disturbances (using (90)) is

$$c_I k \approx \frac{3\beta_0 U_\infty^3 k^2}{2g}.$$

If we use the experimental parameters given above, we find that the growth rate is approximately β_0. This suggests that the void waves do not have time to grow significantly in normal experimental settings. Therefore our model predicts that the observed void waves should grow slightly and travel with a phase speed given by c_R. Recent experiments of Zenit, Koch, and Sangani [58] demonstrate a small amount of clustering.

Therefore, it appears that our model provides an accurate description of long wavelength disturbances. It is possible that the model breaks down for small wavelength disturbances and that there are regularizing effects which, when included in our model, will result in a well-posed model. Another possibility is that Sangani and Didwania [39] and Smereka [42] overestimate the amount of cluster formation observed in experiments. This could be because in these numerical simulations the computational domain was a cube with a size of only a few centimeters. Thus they were exciting modes of a much smaller wavelength than those observed in experiments, due to the periodic boundary conditions.

Finally, we remark that, if we assume the wave frequencies and wave numbers are small, then it follows from (84) that

$$g - \frac{gU}{U_\infty}\left(\frac{(1+\beta+\beta^2)}{1-\beta}\right) \approx 0,$$

which implies that

$$U \approx U_0(\beta),$$

where U_0 is given by (85). We can rewrite (83) using the above expression as

$$\frac{\partial \beta}{\partial t} + (U_0(\beta) + \beta U_0'(\beta))\frac{\partial \beta}{\partial x} \approx 0.$$

This equation was obtained by Lammers and Biesheuvel [26]. Using (85), we note that the above equation can be written as

$$\frac{\partial \beta}{\partial t} + c_R \frac{\partial \beta}{\partial x} \approx 0,$$

where c_R is given by (89). Thus we see, again, that when the frequency and wavenumber of the waves are small, the void wave should travel with a speed given by c_R.

6. Summary and conclusions. In this paper we have developed a new method for solving Laplace's equation for the velocity potential in a liquid with a finite number of bubbles. This method is a generalization of the method of images. Our approach also allows us to define the ambient velocity and ambient pressure associated with a particular bubble. The velocity potential is then used to calculate the total kinetic energy of the liquid. We then use the Euler–Lagrange equation to compute exact equations of motion for a finite collection of bubbles, which are a set of ordinary differential equations. We then make a simplifying approximation, which is to keep only terms arising from monopoles and dipoles. We then take the continuum limit of the equations of motion to obtain a set of partial differential equations that represent our effective equations for ideal bubbly fluids. Our model includes both sound and void wave propagation, includes nonlinear effects, and is valid over a wide range of wave numbers. We show that our model captures the results for the speed of sound waves from Caflisch et al. [10], Crespo [12], and Sangani [38]. We also show that our model reduces to Geurst's model [17, 18] when we consider void wave propagation.

We then consider the effects of liquid viscosity and gravity on void wave propagation. The effects of liquid viscosity are incorporated by using an energy dissipation function. We apply this technique for finite collection of bubbles, thus extending the work of van Wijngaarden and Kapteyn [52] for the two-bubble problem. We then compute the drag force. The continuum limit of the drag force is found, and our effective equations for void waves including gravity and liquid viscosity are formulated. We then observe that our model has a steady-state solution which corresponds to a mixture of bubbles rising with a steady speed. The calculated bubble rise speed is in good agreement with experimental values. We also compute the speed of void waves and find good agreement with experimental results. Our computations show that these waves are unstable, but, using the experimental parameters, we find that the instability can be small.

Appendix A. Proof of Theorem 3.1. Theorem 3.1 states that the method of images, when generalized to N spheres, results in a converging sequence which is the solution to Laplace's equation with the correct boundary conditions.

We begin our proof with some definitions. Let $B = B(\mathbf{p}, R)$ be a ball of radius R centered at the point $\mathbf{p}$. We define the energy norms:

$$||f||_B = \int_B |\nabla f|^2 d\mathbf{x} \quad \text{and} \quad ||f||_{\overline{B}} = \int_{\overline{B}} |\nabla f|^2 d\mathbf{x},$$

where $\overline{B}$ is the region exterior to B. We will also use the L_2 norm on the surface of B (∂B):

$$||f||_{\partial B} = \left(\int_{\partial B} f^2 ds \right)^{\frac{1}{2}}.$$

We shall make use of Weiss' sphere theorem, which, in the notation of our paper, is the following: *If $f(\mathbf{x})$ is harmonic inside $B = B(\mathbf{p}, R)$, then the image operator with respect to B is*

$$I_B f(r, \theta, \psi) = \frac{1}{R} \int_0^{\frac{R^2}{r}} w \frac{\partial f}{\partial w}(w, \theta, \psi) dw.$$

This can be found, for example, in Milne-Thompson [30, p. 520].

Remark. If the closest singularity of f has distance d from $\mathbf{p}$, then $I_B f$ is harmonic for all points outside of $B^*(\mathbf{p}, \frac{R^2}{d})$, which is a sphere smaller than $B(\mathbf{p}, R)$.

Our proof begins with the following three lemmas.

LEMMA A.1. *$B^m = B^m(\mathbf{p}, mR)$, $B = B(\mathbf{p}, R)$, and $B^M = B^M(\mathbf{p}, MR)$ are three concentric spheres with*

$$m < 1 < M, \quad c < 1, \quad 1 < cMm;$$

f is a harmonic function inside B^M; and $I_B f$ is harmonic exterior to B^m. Then we have

$$||I_B f||_{\overline{B^m}} < c||f||_{B^M},$$

where $\overline{B^m}$ is the region exterior to B^m.

Proof. Assume that $\mathbf{p}$ is at the origin. We can write f in terms of spherical harmonics:

$$f(r,\theta,\psi) = f(0) + \sum_{k=1}^{\infty} \sum_{j=1}^{2k+1} c_{k,j} h_{k,j}(\theta,\psi) r^k,$$

where $c_{k,j}$ are constants and $h_{k,j}$ is a set of spherical harmonics which satisfy the orthogonality condition

$$\int_{\text{unit ball}} h_{k,j} h_{m,i} dS = \delta_{km} \delta_{ij}.$$

From Weiss' sphere theorem, we have

$$\begin{aligned} I_B f &= \frac{1}{R} \int_0^{\frac{R^2}{r}} w \frac{\partial f}{\partial w}(w,\theta,\psi) dw \\ &= \frac{1}{R} \sum_{k=1}^{\infty} \sum_{j=1}^{2k+1} \frac{k}{k+1} c_{k,j} h_{k,j} R^{k+1} \Big|_0^{\frac{R^2}{r}} \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{2k+1} \frac{k R^{2k+1}}{(k+1) r^{k+1}} c_{k,j} h_{k,j}. \end{aligned}$$

We can use the divergence theorem and the expansion of f in spherical harmonics to obtain

$$||f||_{B^M} = \int_{\partial B^M} f \frac{\partial f}{\partial n} ds = \sum_{k=1}^{\infty} \sum_{j=1}^{2k+1} z_{k,j}^{(1)} \tag{91}$$

with

$$z_{k,j}^{(1)} = k (MR)^{2k+1} c_{k,j}^2.$$

In a similar fashion,

$$||I_B f||_{\overline{B^m}} = - \int_{\partial B^m} I_B f \frac{\partial I_B f}{\partial n} ds = \sum_{k=1}^{\infty} \sum_{j=1}^{2k+1} z_{k,j}^{(2)}, \tag{92}$$

where

$$z_{k,j}^{(2)} = \frac{k^2 R^{4k+2}}{(k+1)(mR)^{2k+1}} c_{k,j}^2.$$

Therefore one has

$$\frac{z_{k,j}^{(2)}}{z_{k,j}^{(1)}} = \frac{k}{k+1} \left(\frac{1}{M^{2k+1} m^{2k+1}} \right).$$

Since $cmM < 1$, it follows that

$$\frac{z_{k,j}^{(2)}}{z_{k,j}^{(1)}} < c.$$

Hence

$$||I_B f||_{\overline{B^m}} < c||f||_{B^M}.$$

This completes the proof of Lemma A.1.

LEMMA A.2. *If f is harmonic outside $B(p, R)$, then*

$$||f||^2_{\partial B} \leq R||f||_{\overline{B}},$$

where $\overline{B}$ is the region outside of B.

Proof. After expanding

$$f(r, \theta, \psi) = \sum_{k=1}^{\infty} \sum_{m=1}^{2k-1} \frac{c_{k,m} h_{k,m}(\theta, \psi)}{r^{k+1}},$$

we have

$$||f||^2_{\partial B} = \sum_{k=1}^{\infty} \sum_{m=1}^{2k-1} \frac{c^2_{k,m}}{R^{2k}} \leq R \sum_{k=1}^{\infty} \sum_{m=1}^{2k-1} \frac{k c^2_{k,m}}{R^{2k+1}} = R||f||_{\overline{B}}.$$

This completes the proof of Lemma A.2.

LEMMA A.3. *If $f_k, k = 1, 2, \ldots,$ are harmonic functions outside $B(\mathbf{p}, R)$ and*

$$||f_k||_{\overline{B}} < Y c^k, \qquad c < 1,\ Y \text{ are constants},$$

then there exists a function f such that, at all points $\mathbf{x}$ outside B, f is harmonic and

$$\lim_{k\to\infty} \sum_{j=1}^{k} f_j(\mathbf{x}) = f(\mathbf{x}),$$

$$\lim_{\mathbf{x}\to\infty} f(\mathbf{x}) = 0.$$

Furthermore, the convergence is uniform outside $B^*(\mathbf{p}, R^*)$ for any $R^* > R$.

Proof. We assume that $\mathbf{p}$ is at the origin. From Lemma A.2, we have

$$||f_k||^2_{\partial B} < R||f_k||_{\overline{B}} < RY c^k.$$

Hence

$$||f_k||_{\partial B} < \sqrt{RY}(\sqrt{c})^k.$$

Therefore $\{\sum_{i=1}^k f_i\}_k$ is a Cauchy sequence in the $||\cdot||_{\partial B}$ norm. Since the L^2 space on ∂B is complete, there exists a function $f \in L^2(\partial B)$ such that

$$\lim_{k\to\infty} \left\| \sum_{i=1}^{k} f_i - f \right\|_{\partial B} = 0.$$

We define f, for any $\mathbf{x}$ outside B, by using

$$f(\mathbf{x}) = \int_{\partial B} P(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) dS_{\mathbf{y}}, \tag{93}$$

where $P(\mathbf{x}, \mathbf{y})$ is the Poisson kernel. From Axler, Bourdon, and Ramey [2], we have

$$P(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi R} \frac{|\mathbf{x}|^2 - R^2}{|\mathbf{x} - \mathbf{y}|^3}.$$

Therefore

$$\begin{aligned} \left| f(\mathbf{x}) - \sum_{i=1}^{k} f_i(\mathbf{x}) \right| &= \left| \int_{\partial B} P(\mathbf{x}, \mathbf{y}) \left(f(\mathbf{y}) - \sum_{i=1}^{k} f_i(\mathbf{y}) \right) dS_{\mathbf{y}} \right| \\ &\leq ||P(\mathbf{x}, \mathbf{y})||_{\partial B} \cdot \left\| f - \sum_{i=1}^{k} f_i \right\|_{\partial B}, \end{aligned}$$

where the last inequality comes from the Cauchy–Schwarz inequality We also have a uniform bound on P for all $\mathbf{y} \in \partial B$, when $\mathbf{x}$ is a fixed point strictly outside B,

$$P(\mathbf{x}, \mathbf{y}) \leq \frac{1}{4\pi R} \frac{|\mathbf{x}|^2 - R^2}{(|\mathbf{x}| - |\mathbf{y}|)^3} = \frac{1}{4\pi R} \frac{|\mathbf{x}| + R}{(|\mathbf{x}| - R)^2}.$$

Hence

$$\lim_{k \to \infty} \left| f(\mathbf{x}) - \sum_{i=1}^{k} f_i(\mathbf{x}) \right| = 0. \tag{94}$$

Furthermore, if $|\mathbf{x}| \geq R^* > R$, then

$$P(\mathbf{x}, \mathbf{y}) \leq \frac{1}{4\pi R} \frac{R^* + R}{(R^* - R)^2}.$$

Thus the convergence in (94) is uniform for $|\mathbf{x}| \geq R^*$.

In this context, f is harmonic because, from (93) and the fact that $P(\mathbf{x}, \mathbf{y})$ is harmonic in $\mathbf{x}$ if $\mathbf{y}$ is fixed,

$$\triangle f = \int_{\partial B} \triangle_{\mathbf{x}} P(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) dS_{\mathbf{y}} = 0.$$

Since $P(\mathbf{x}, \mathbf{y})$ vanishes when $\mathbf{x} \to \infty$, then so does f because of (93). This completes the proof of Lemma A.3.

Proof of Theorem 3.1. Since all the spheres $B_i(\mathbf{x}_i, R_i)$, $i = 1, \dots, N$, do not intersect and from the remark after the statement of Weiss' sphere theorem, we conclude that $I_i I_j \cdots \phi_k$ is harmonic not only outside $B_i(\mathbf{x}_i, R_i)$, but also outside a smaller sphere $B^*(\mathbf{x}_i, R_i^2/d_{ij})$, where d_{ij} is the distance between $\mathbf{x}_i$ and the closest point on ∂B_j because $I_j \cdots \phi_k$ has singularities only inside B_j. Therefore, we can find constant m, M, c such that

- $m < 1 < M$, $c < 1$, $1 < cMm$,
- $B_i^M(p_i, MR_i)$ do not intersect,
- $I_i \cdots I_j \phi_k$ is harmonic outside $B_i^m(p_i, mR_i)$.

To achieve this we let 2ε be the smallest distance between the surface of any of the N spheres. Let $q = \min_i \frac{R_i + \varepsilon}{R_i}$; then the choice $M = q^{\frac{3}{4}}$, $m = q^{-\frac{1}{4}}$, and $c = m$ will work (for example).

Next, we write

$$\phi = \sum_{j=1}^{N} \Omega_j,$$

where

$$\Omega_j = \phi_j + \sum_{\substack{i=i_1\\ i_1\neq j}}^{N} I_j\phi_i + \cdots + \sum_{\substack{i_1,\ldots,i_k=1\\ i_1\neq j, i_\ell\neq i_{\ell+1}}}^{N} I_j I_{i_1} I_{i_2}\cdots I_{i_{k-1}}\phi_{i_k}\cdots. \tag{95}$$

From the remarks above we know that Ω_j is harmonic outside $B_j^m(\mathbf{x}_j, mR_j)$.

Our plan is to prove that each term of Ω_j satisfies an estimate of the form given in Lemma A.3. Therefore we must estimate each term in the series. We denote the kth term of (95) as T_k and let

$$Y = \sum_{i=1}^{N} ||\phi_i||_{\overline{B_i^m}}.$$

Then it is obvious that

$$||T_0||_{\overline{B_j^m}} = ||\phi_j||_{\overline{B_j^m}} < Y. \tag{96}$$

To estimate the second term we appeal to Lemma A.1, and we have for $i_1 \neq j$

$$||I_j\phi_{i_1}||_{\overline{B_j^m}} < c||\phi_{i_1}||_{B_j^M} < c||\phi_{i_1}||_{\overline{B_{i_1}^m}}.$$

The second inequality follows since B_j^M is contained in $\overline{B_{i_1}^m}$. Now, we sum the above inequality over i_1 to obtain

$$||T_1||_{\overline{B_j^m}} < c\sum_{\substack{i=i_1\\ i_1\neq j}}^{N} ||\phi_{i_1}||_{\overline{B_{i_1}^m}} < cY. \tag{97}$$

Now, we look at the kth term and use Lemma A.1 to obtain the following estimate:

$$||T_k||_{\overline{B_j^m}} < c\sum_{\substack{i_1,\ldots,i_k=1\\ i_1\neq j, i_\ell\neq i_{\ell+1}}}^{N} ||I_{i_1}I_{i_2}\cdots I_{i_{k-1}}\phi_{i_k}||_{B_j^M}.$$

Next we use the above estimate and the fact that B_j^M is contained in $\overline{B_{i_1}^m}$ for $i_1 \neq j$ to find

$$||T_k||_{\overline{B_j^m}} < c\sum_{\substack{i_1,\ldots,i_k=1\\ i_1\neq j, i_\ell\neq i_{\ell+1}}}^{N} ||I_{i_1}I_{i_2}\cdots I_{i_{k-1}}\phi_{i_k}||_{\overline{B_{i_1}^m}}. \tag{98}$$

Applying Lemma A.1 again, we find

$$||T_k||_{\overline{B_j^m}} < c^2\sum_{\substack{i_1,\ldots,i_k=1\\ i_1\neq j, i_\ell\neq i_{\ell+1}}}^{N} ||I_{i_2}\cdots I_{i_{k-1}}\phi_{i_k}||_{B_{i_1}^M}. \tag{99}$$

Since $i_1 \neq i_2$, it then follows that all $B_{i_1}^M$ are contained in $\overline{B_{i_2}^m}$, and we find from the above inequality

$$||T_k||_{\overline{B_j^m}} < c^2\sum_{\substack{i_2,\ldots,i_k=1\\ i_\ell\neq i_{\ell+1}}}^{N} ||I_{i_2}\cdots I_{i_{k-1}}\phi_{i_k}||_{\overline{B_{i_2}^m}}. \tag{100}$$

This is of the same form as (98), so therefore we repeat the same steps as were used to obtain (99) and (100) and thereby obtain the estimate

$$||T_k||_{\overline{B_j^m}} < c^k Y.$$

Next we combine Lemma A.2 and Lemma A.3 to conclude that Ω_j is harmonic and uniformly convergent outside bubble j. It then follows that ϕ is also harmonic and uniformly convergent in the liquid region. The completes our proof of Theorem 3.1.

Appendix B. Proof of Theorem 3.2. We begin with the following formula. Suppose $\mathbf{r} = (x_1, x_2, x_3)$; then we have

(101)
$$(-1)^n \left(\nabla^n \left(\frac{1}{|\mathbf{r}|}\right)\right)_{i_1 \cdots i_n} = \frac{(2n-1)!! x_{i_1} \cdots x_{i_n}}{|\mathbf{r}|^{2n+1}} + \sum_{j=1}^{N_2} \frac{(-1)^j (2n-2j-1)!! A_j}{|\mathbf{r}|^{2n-2j+1}},$$

with $i_1, i_2, \ldots, i_n = 1, 2, 3$; N_2 is the integer part of $\frac{n}{2}$; and

$$A_j = \sum \delta_{k_1 k_2} \cdots \delta_{k_{2j-1} k_{2j}} x_{k_{2j+1}} \cdots x_{k_n},$$

where the sum is over all possible j pairs $(k_1, k_2) \cdots (k_{2j-1}, k_{2j})$ from i_1 to i_n.

This formula can be proven by induction. Some examples of (101) are

$$\begin{aligned}
\left(\nabla \left(\frac{1}{|\mathbf{r}|}\right)\right)_i &= -\frac{x_i}{|\mathbf{r}|^3}, \\
\left(\nabla^2 \left(\frac{1}{|\mathbf{r}|}\right)\right)_{ij} &= \frac{3x_i x_j}{|\mathbf{r}|^5} - \frac{\delta_{ij}}{|\mathbf{r}|^3}, \\
\left(\nabla^3 \left(\frac{1}{|\mathbf{r}|}\right)\right)_{ijk} &= \frac{-15 x_i x_j x_k}{|\mathbf{r}|^7} + \frac{3(\delta_{ij} x_k + \delta_{ik} x_j + \delta_{jk} x_i)}{|\mathbf{r}|^5},
\end{aligned}$$

where i, j, and k run from 1 to 3.

One can use (101) to prove the following:

(102)
$$\frac{(-1)^k}{(2k-1)!!} \nabla^k f \cdot \nabla^k \left(\frac{1}{|\mathbf{r}|}\right) = \nabla^k f \cdot \frac{\mathbf{n}^k}{|\mathbf{r}|^{k+1}},$$

where $k \geq 2$ and

$$\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}.$$

To prove (102) is straightforward. We first notice, since f is harmonic, that

$$\nabla^k f \cdot \delta_{ij} = 0.$$

Combining this result with (101), we obtain (102). We are now ready to prove Theorem 3.2; without losing the generality, we assume that $\mathbf{p}$ is the origin. According to Weiss's sphere theorem, we have

$$I_B f(\mathbf{r}) = \frac{1}{R} \int_0^{\frac{R^2}{|\mathbf{r}|}} w \nabla f (w\mathbf{n}) \cdot \mathbf{n} dw.$$

Using a Taylor series expansion for $f(\mathbf{r})$, we can rewrite the above expression as

$$I_B f(\mathbf{r}) = \frac{1}{R}\int_0^{\frac{R^2}{|\mathbf{r}|}} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} w^k \nabla^k f(0)\cdot \mathbf{n}^k dw.$$

Integrating term by term, we obtain

$$I_B f(\mathbf{r}) = \sum_{k=1}^{\infty} \frac{R^{2k+1}}{(k-1)!(k+1)|\mathbf{r}|^{k+1}} \nabla^k f(0)\cdot \mathbf{n}^k. \tag{103}$$

Next we combine (102) with (103) to obtain Theorem 3.2. This completes the proof of Theorem 3.2.

Appendix C. Useful formulas. If $g(\mathbf{x})$ is a harmonic function in $B(\mathbf{p}, R)$ and $\mathbf{d}$, $\mathbf{e}$ are constant vectors, then

$$\int_{\partial B} g(\mathbf{x})ds = 4\pi R^2 g(\mathbf{p}),$$

$$\int_{\partial B} (\mathbf{d}\cdot\mathbf{n}) g(\mathbf{x})ds = \frac{4\pi R^3}{3}\mathbf{d}\cdot\nabla g(\mathbf{p}),$$

$$\int_{\partial B} (\mathbf{d}\cdot\mathbf{n})(\mathbf{e}\cdot\mathbf{n})ds = \frac{4\pi}{3}R^2\mathbf{d}\cdot\mathbf{e},$$

$$\int_{\partial B} I_B g(\mathbf{x})ds = 0,$$

$$\text{and} \quad \int_{\partial B} (\mathbf{d}\cdot\mathbf{n})\, I_B g(\mathbf{x})ds = \frac{2\pi R^3}{3}\mathbf{d}\cdot\nabla g(\mathbf{p}).$$

These formulas are established using the orthogonality properties of spherical harmonics. Theorem 3.2 is used in the proof of the last two equations.

Appendix D. Derivative calculations. In this section we shall compute the derivatives of K that appear in the Euler–Lagrange equations. When computing these derivatives, we will not use the expression for K given by (20). Instead we will use the integral form of K, which we rewrite here:

$$K = -\frac{1}{2}\rho_\ell \sum_{j=1}^{N} \int_{S_j} \phi \frac{\partial \phi}{\partial n} ds. \tag{104}$$

We have found it useful, when computing these derivatives, to introduce the operator J_i, which is defined as follows: if we have N spheres $B_1, B_2, \ldots, B_N$ and a function $f(\mathbf{x})$, which is harmonic outside of B_i, then we say that

$$g(\mathbf{x}) = J_i f(\mathbf{x})$$

if

$$\begin{aligned} \frac{\partial g}{\partial n} &= \frac{\partial f}{\partial n} \quad \text{at } \partial B_i, \\ \frac{\partial g}{\partial n} &= 0 \qquad \text{at } \partial B_j \text{ when } j \neq i. \end{aligned} \tag{105}$$

We note that $g(\mathbf{x})$ will be harmonic outside all of the spheres.

The operator can be expressed in terms of the image operator (defined in Theorem 3.1) as follows:

$$J_i(f) = f + \sum_{i_1=1, i_1\neq i}^{N} I_{i_1} f + \cdots + \sum_{i_1,\ldots,i_k=1, i_k\neq i, i_j\neq i_{j+1}}^{N} I_{i_1} I_{i_2} \cdots I_{i_k} f + \cdots. \tag{106}$$

This can be seen by applying Theorem 3.1 with $\phi_i = f$ and $\phi_j = 0$ if $j \neq i$.

It is easy to verify that the solution to (11) is

$$\phi = \sum_{i=1}^{N} J_i \phi_i, \tag{107}$$

where ϕ_i is defined in Theorem 3.1.

One property associated with operator J that we will use, if f and g are harmonic, is

$$\int_S J_i f \frac{\partial g}{\partial n} ds = \int_S \frac{\partial J_i f}{\partial n} g ds = \int_{S_i} \frac{\partial f}{\partial n} g ds, \tag{108}$$

where the first equality is Green's theorem and the second equality follows from the definition of the operator J.

Proof of (22). We have from (104)

$$\begin{aligned} K &= -\frac{\rho_\ell}{2} \sum_{j=1}^{N} \int_{S_j} \phi \frac{\partial \phi}{\partial n} ds \\ &= -\frac{\rho_\ell}{2} \rho_\ell \sum_{j=1}^{N} \int_{S_j} \phi (\dot{R}_j + \mathbf{U}_j \cdot \mathbf{n}) ds. \end{aligned}$$

Thus we have

$$-\frac{2}{\rho_\ell} \frac{\partial K}{\partial \dot{R}_i} = \int_{S_i} \phi ds + \sum_{j=1}^{N} \int_{S_j} \frac{\partial \phi}{\partial \dot{R}_i} \frac{\partial \phi}{\partial n} ds. \tag{109}$$

We will consider the two terms on the right-hand side of (109) separately. One has

$$\int_{S_i} \phi ds = \int_{S_i} (\phi_i + \psi_i + I_i \psi_i) ds.$$

It follows from Appendix C that this becomes

$$\int_{S_i} \phi ds = 4\pi R_i^2 (-R_i \dot{R}_i + \psi_i(\mathbf{x}_i)). \tag{110}$$

Turning to the second term, we have

$$\begin{aligned} \sum_{j=1}^{N} \int_{S_j} \frac{\partial \phi}{\partial \dot{R}_i} \frac{\partial \phi}{\partial n} ds &= \int_S \frac{\partial \phi}{\partial \dot{R}_i} \frac{\partial \phi}{\partial n} ds \\ &= \sum_{j=1}^{N} \int_S \frac{\partial J_j \phi_j}{\partial \dot{R}_i} \frac{\partial \phi}{\partial n} ds, \end{aligned}$$

where we have used (107). Next we observe that $J_j\phi_j$ does not depend on $\dot{R}_i$ unless $i=j$; thus we have

$$\sum_{j=1}^{N}\int_{S_j}\frac{\partial\phi}{\partial\dot{R}_i}\frac{\partial\phi}{\partial n}ds=\int_S\frac{\partial J_i\phi_i}{\partial\dot{R}_i}\frac{\partial\phi}{\partial n}ds.$$

Furthermore, the operator J_i does not depend on $\dot{R}_i$; thus the previous expression becomes

$$\sum_{j=1}^{N}\int_{S_j}\frac{\partial\phi}{\partial\dot{R}_i}\frac{\partial\phi}{\partial n}ds=\int_S J_i\left(\frac{\partial\phi_i}{\partial\dot{R}_i}\right)\frac{\partial\phi}{\partial n}ds. \tag{111}$$

Applying (108), we find

$$\int_S J_i\left(\frac{\partial\phi_i}{\partial\dot{R}_i}\right)\frac{\partial\phi}{\partial n}ds=\int_{S_i}\frac{\partial}{\partial n}\left(\frac{\partial\phi_i}{\partial\dot{R}_i}\right)\phi ds=\int_{S_i}\phi ds. \tag{112}$$

Combining (109), (110), (111), and (112), we find

$$\frac{\partial K}{\partial\dot{R}_i}=2\pi\rho_\ell(2R_i^3\dot{R}-2R_i^2\psi_i(\mathbf{x}_i)).$$

Thus (22) is proven. The proof for (23) is similar to that above.

Proof of (24). When calculating $\frac{\partial K}{\partial R_i}$, we notice that R_i enters into K in (104) in three ways:

- the integration region depends on R_i,
- ϕ_i depends on R_i,
- the image potential operator I_i depends on R_i.

Therefore we have

$$\frac{\partial K}{\partial R_i}=\frac{\partial K}{\partial R_i^S}+\frac{\partial K}{\partial R_i^\phi}+\frac{\partial K}{\partial R_i^I},$$

where R_i^S, R_i^ϕ, and R_i^I represent R_i in the integration region, in ϕ_i, and in I_i, respectively.

Step 1. We first assume that ϕ_i and I_i are fixed and consider only the effect of changing the integration region. We have

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^S}=\frac{\partial}{\partial R_i^S}\int_S\phi\frac{\partial\phi}{\partial n}ds=\frac{\partial}{\partial R_i^S}\int_{S_i}\phi\frac{\partial\phi}{\partial n}ds.$$

It follows from the above result and (17) that

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^S}=\frac{\partial}{\partial R_i^S}\int_{S_i}(\phi_i+\psi_i+I_i\psi_i)\frac{\partial\phi}{\partial n}ds.$$

We continue our calculation by applying Theorem 3.2 to expand $I_i\psi_i$ and using a Taylor expansion for ψ_i to obtain

$$\begin{aligned}-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^S}=\frac{\partial}{\partial R_i^S}\int_{S_i}\Bigg[&-\frac{R_i^2\dot{R}_i}{|\mathbf{r}-\mathbf{x}_i|}+\frac{1}{2}R_i^3\nabla_r\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\cdot\mathbf{U}_i\\&+\psi_i(\mathbf{x}_i)+\mathbf{v}_i(\mathbf{x}_i)\cdot(\mathbf{r}-\mathbf{x}_i)-\frac{1}{2}R_i^3\mathbf{v}_i(\mathbf{x}_i)\cdot\nabla_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\\&+\text{(higher order harmonics)}\Bigg](\dot{R}_i+\mathbf{U}_i\cdot\mathbf{n})ds.\end{aligned}$$

Evaluating the integrand on S_i ($|\mathbf{r}| = R_i$) and using the orthogonality properties of spherical harmonics, we find

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^S} = \frac{\partial}{\partial R_i^S}\int_{S_i}\Bigg[-\frac{R_i^2\dot{R}_i}{R_i^S} - \frac{R_i^3}{2(R_i^S)^2}\mathbf{U}_i\cdot\mathbf{n} + \psi_i(\mathbf{x}_i) + R_i^S\mathbf{v}_i(\mathbf{x}_i)\cdot\mathbf{n} + \frac{R_i^3}{2(R_i^S)^2}\mathbf{v}_i(\mathbf{x}_i)\cdot\mathbf{n}\Bigg](\dot{R}_i + \mathbf{U}_i\cdot\mathbf{n})ds.$$

The integration over S_i is performed using the results in Appendix C, and we obtain

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^S} = \frac{\partial}{\partial R_i^S}\Bigg[4\pi(R_i^S)^2\left(-\frac{R_i^2\dot{R}_i^2}{R_i^S} + \psi_i(\mathbf{x}_i)\dot{R}_i\right) + \frac{4\pi(R_i^S)^2}{3}\mathbf{U}_i\cdot\left(-\frac{R_i^3\mathbf{U}_i}{2(R_i^S)^2} + R_i^S\mathbf{v}_i(\mathbf{x}_i) + \frac{R_i^3}{2(R_i^S)^2}\mathbf{v}_i(\mathbf{x}_i)\right)\Bigg].$$

Next, we take the derivative of the above expression and evaluate it at $R_i^S = R_i$ to obtain

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^S} = -4\pi R_i^2\dot{R}_i^2 + 8\pi R_i\dot{R}_i\psi_i(\mathbf{x}_i) + 4\pi R_i^2\mathbf{U}_i\cdot\mathbf{v}_i(\mathbf{x}_i). \tag{113}$$

Step 2. We now assume that the integration region and I_i are fixed, and that only R_i in ϕ_i is changing. We need to calculate

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^\phi} = \frac{\partial}{\partial R_i^\phi}\int_S \phi\frac{\partial\phi}{\partial n}ds.$$

Applying (107) and using the fact that only ϕ_i is changing, we have

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^\phi} = \frac{\partial}{\partial R_i^\phi}\int_S J_i(\phi_i)\frac{\partial\phi}{\partial n}ds.$$

In this case neither J_i nor $\frac{\partial\phi}{\partial n}$ depend on R_i^ϕ; thus we may write the above equation as

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^\phi} = \int_S J_i\left(\frac{\partial\phi_i}{\partial R_i^\phi}\right)\frac{\partial\phi}{\partial n}ds.$$

Substituting our expression for ϕ_i given by (13), we have

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^\phi} = \int_S J_i\left(-\frac{2R_i^\phi\dot{R}_i}{|\mathbf{r}-\mathbf{x}_i|} + \frac{3}{2}(R_i^\phi)^2\nabla_r\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\cdot\mathbf{U}_i\right)\frac{\partial\phi}{\partial n}ds.$$

It follows from (108) and (17) that we can write the previous expression as

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^\phi} = \int_{S_i}\frac{\partial}{\partial n}\left(-\frac{2R_i^\phi\dot{R}_i}{|\mathbf{r}-\mathbf{x}_i|} + \frac{3}{2}(R_i^\phi)^2\nabla_r\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\cdot\mathbf{U}_i\right)(\phi_i + \psi_i + I_i\psi_i)ds.$$

Evaluating the normal derivatives at the bubble surface and setting $R_i^{\phi} = R_i$, we obtain

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^{\phi}} = \int_{S_i}\left(\frac{2\dot{R}_i}{R_i} + \frac{3\mathbf{U}_i\cdot\mathbf{n}}{R_i}\right)(\phi_i + \psi_i + I_i\psi_i)ds.$$

Next, (13) and the results of Appendix C are used to deduce

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^{\phi}} = -8\pi R_i^2\dot{R}_i^2 + 8\pi R_i\dot{R}_i\psi_i(\mathbf{x}_i) - 2\pi R_i^2|\mathbf{U}_i|^2 + 6\pi R_i^2\mathbf{U}_i\cdot\mathbf{v}_i(\mathbf{x}_i). \tag{114}$$

Step 3. Finally, we wish to calculate $\frac{\partial K}{\partial R_i}$, where only R_i in the operator I_i is changing. We start with

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^{I}} = \frac{\partial}{\partial R_i^{I}}\int_S \phi\frac{\partial\phi}{\partial n}ds.$$

Since $\frac{\partial\phi}{\partial n}$ and the region of integration does not depend on R_i^I, we then have

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^{I}} = \int_S \frac{\partial\phi}{\partial R_i^{I}}\frac{\partial\phi}{\partial n}ds. \tag{115}$$

The task at hand then is to calculate $\frac{\partial\phi}{\partial R_i^I}$. This will be done using (14). Some terms of ϕ in (14) have multiple occurrences of I_i. For these terms, we separate each into several terms, so that after the separation, each term only has one changing I_i, denoted by $\tilde{I}_i$. For example, we have

$$\frac{\partial}{\partial R_1^I}I_1I_2I_1I_3I_4I_1\phi_3 = \frac{\partial}{\partial R_1^I}\left(\tilde{I}_1I_2I_1I_3I_4I_1\phi_3 + I_1I_2\tilde{I}_1I_3I_4I_1\phi_3 + I_1I_2I_1I_3I_4\tilde{I}_1\phi_3\right).$$

By doing this, we find

$$\frac{\partial\phi}{\partial R_i^I} = \frac{\partial}{\partial R_i^I}\left(\left(Id + \sum_{j_l\neq j_{l+1},j_m\neq i} I_{j_1}\cdots I_{j_m}\right)\tilde{I}_i\left(\sum_{i\neq k_1,k_l\neq k_{l+1}} I_{k_1}\cdots I_{k_n}\phi_{k_{n+1}}\right)\right),$$

where Id is the identity operator. From the expressions of J_i and ψ_i in (106) and (16), we obtain

$$\frac{\partial\phi}{\partial R_i^I} = \frac{\partial J_i\tilde{I}_i\psi_i}{\partial R_i^I}.$$

Since J_i does not depend on R_i^I, we can write the previous equation as

$$\frac{\partial\phi}{\partial R_i^I} = J_i\left(\frac{\partial\tilde{I}_i\psi_i}{\partial R_i^I}\right).$$

We use the above equation to rewrite (115) as

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^{I}} = \int_S J_i\left(\frac{\partial\tilde{I}_i\psi_i}{\partial R_i^I}\right)\frac{\partial\phi}{\partial n}ds.$$

It follows from (108) that

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^I} = \int_{S_i} \phi \frac{\partial}{\partial n}\left(\frac{\partial \tilde{I}_i \psi_i}{\partial R_i^I}\right) ds$$
$$= \int_{S_i} (\phi_i + \psi_i + I_i\psi_i) \frac{\partial}{\partial n}\left(\frac{\partial \tilde{I}_i \psi_i}{\partial R_i^I}\right) ds.$$

If we expand $\tilde{I}_i\psi_i$ using (103), we can show

$$\frac{\partial}{\partial n}\left(\frac{\partial \tilde{I}_i \psi_i}{\partial R_i^I}\right) = W = \sum_{k=1}^{\infty} \frac{-(2k+1)(R_i^I)^{k-2}}{(k-1)!} \nabla^k \psi_i(\mathbf{x}_i) \cdot \mathbf{n}^k.$$

We also expand $I_i\psi_i$ using (103) and expand ψ_i in a Taylor series to obtain

$$-\frac{2}{\rho_\ell}\frac{\partial K}{\partial R_i^I} = \int_{S_i} \Bigg[- R_i\dot{R}_i - \frac{R_i}{2}\mathbf{U}_i \cdot \mathbf{n} + \sum_{k=0}^{\infty} \frac{1}{k!} R_i^k \nabla^k \psi_i(\mathbf{x}_i) \cdot \mathbf{n}^k$$
$$+ \sum_{k=1}^{\infty} \frac{R_i^{2k+1}}{(k-1)!(k+1)R_i^{k+1}} \nabla^k \psi_i(\mathbf{x}_i) \cdot \mathbf{n}^k \Bigg] W ds$$

$$= \int_{S_i} \left(-\frac{R_i}{2}\mathbf{U}_i \cdot \mathbf{n} + \sum_{k=1}^{\infty} \frac{2k+1}{(k+1)!} R_i^k \nabla^k \psi_i(\mathbf{x}_i) \cdot \mathbf{n}^k \right) W ds$$

$$= 2\pi R_i^2 \mathbf{U}_i \cdot \mathbf{v}_i(\mathbf{x}_i) - 6\pi R_i^2 |\mathbf{v}_i(\mathbf{x}_i)|^2 + F, \tag{116}$$

where we have let $R_i^I = R_i$. F contains terms involving $\nabla^k\psi \cdot \nabla^k\psi$ with $k > 1$. We have used the orthogonality of spherical harmonics and the results in Appendix C to arrive at (116). Combining all three parts ((113), (114), and (116)), we have

$$\frac{\partial K}{\partial R_i} = 2\pi\rho_\ell \left(3R_i^2\dot{R}_i^2 + \frac{1}{2}R_i^2|\mathbf{U}_i|^2 - 4R_i\dot{R}_i\psi_i(\mathbf{x}_i) - 3R_i^2\mathbf{U}_i \cdot \mathbf{v}_i(\mathbf{x}_i) + \frac{3}{2}R_i^2|\mathbf{v}_i|^2 + F \right),$$

which is (24).

Proof of (25). We first expand the monopole and dipole terms at $\mathbf{x}$ in Laurent series around $\mathbf{y}$:

$$\frac{1}{|\mathbf{r}-\mathbf{x}|} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{|\mathbf{x}-\mathbf{y}|^{2n+1}}{|\mathbf{r}-\mathbf{y}|^{2n+1}} \nabla_{\mathbf{y}}^n \left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) \cdot (\mathbf{r}-\mathbf{y})^n$$
$$= \frac{1}{|\mathbf{r}-\mathbf{y}|} + \frac{(\mathbf{x}-\mathbf{y}) \cdot (\mathbf{r}-\mathbf{y})}{|\mathbf{r}-\mathbf{y}|^3} + \cdots,$$
$$\nabla_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{x}|}\right) = \sum_{n=0}^{\infty} \frac{-(2n+1)}{n!} \frac{|\mathbf{x}-\mathbf{y}|^{2n+1}}{|\mathbf{r}-\mathbf{y}|^{2n+3}} \nabla_{\mathbf{y}}^n \left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) \cdot (\mathbf{r}-\mathbf{y})^{n+1}$$
$$+ \frac{1}{(n-1)!} \frac{|\mathbf{x}-\mathbf{y}|^{2n+1}}{|\mathbf{r}-\mathbf{y}|^{2n+1}} \nabla_{\mathbf{y}}^n \left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) \cdot (\mathbf{r}-\mathbf{y})^{n-1}$$
$$= \nabla_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{y}|}\right) + \cdots.$$

We follow the same procedure as in the previous section to calculate $\frac{\partial K}{\partial \mathbf{x}_i}$. If $\mathbf{x}_i^I$, $\mathbf{x}_i^\phi$, and $\mathbf{x}_i^I$ are used to represent $\mathbf{x}_i$ in the integration region, in ϕ_i, and in I_i, respectively, we can make our calculation by using the same arguments as in the previous section and the two Laurent series above. Without writing the details, we obtain

$$\frac{\partial}{\partial \mathbf{x}_i^S}\int_S \phi\frac{\partial\phi}{\partial n}ds = 4\pi R_i^2\dot{R}_i\mathbf{v}_i + \frac{4\pi}{3}R_i^2\dot{R}_i\mathbf{U}_i + \frac{4\pi}{3}R_i^2 + (\nabla\mathbf{v}_i)^T\cdot\mathbf{U}_i,$$

$$\frac{\partial}{\partial \mathbf{x}_i^\phi}\int_S \phi\frac{\partial\phi}{\partial n}ds = -\frac{4\pi}{3}R_i^2\dot{R}_i\mathbf{U}_i + 4\pi R_i^2\dot{R}_i\mathbf{v}_i + 2\pi R_i^3(\nabla\mathbf{v}_i)^T\cdot\mathbf{U}_i,$$

$$\frac{\partial}{\partial \mathbf{x}_i^I}\int_S \phi\frac{\partial\phi}{\partial n}ds = \frac{2\pi}{3}R_i^3(\nabla\mathbf{v}_i)^T\cdot\mathbf{U}_i - 4\pi R_i^3 + (\nabla\mathbf{v}_i)^T\cdot\mathbf{v}_i + G,$$

where $\mathbf{v}_i = \mathbf{v}_i(\mathbf{x}_i)$ and G will have only terms involving $\nabla^k\psi_i(\mathbf{x}_i)\cdot\nabla^{k+1}\psi_i(\mathbf{x}_i)$, with $k > 1$. Adding all three parts together, we have

$$\frac{\partial K}{\partial \mathbf{x}_i} = 2\pi\rho_\ell\left(-2R_i^2\dot{R}_i\mathbf{v}_i + R_i^3(\nabla\mathbf{v}_i)^T\cdot(\mathbf{v}_i - \mathbf{U}_i) + G\right).$$

Appendix E. Energy dissipation and drag force. In this appendix, we will calculate energy dissipation of bubbly flow with a finite number of bubbles. We then calculate the drag force on each bubble.

Energy dissipation. From (72), we have

$$\mathcal{D} = -\mu\int_S \frac{\partial(\nabla\phi\cdot\nabla\phi)}{\partial r}ds.$$

Using (17) in the above expression, we have

$$\mathcal{D} = -\mu\sum_{i=1}^N\int_{S_i}\frac{\partial}{\partial n}\left|\nabla\left(\phi_i + \psi_i + I_i\psi_i\right)\right|^2 ds.$$

Applying Theorem 3.2 and expanding ψ_i, we obtain

$$\begin{aligned}\mathcal{D} =&-\mu\sum_{i=1}^N\int_{S_i}\frac{\partial}{\partial n}\Bigg|\nabla\Bigg(-\frac{R_i^2\dot{R}_i}{|\mathbf{r}-\mathbf{x}_i|} + \frac{1}{2}R_i^3\nabla_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\cdot\mathbf{U}_i\\ &\qquad + \psi_i(\mathbf{x}_i) + \mathbf{v}_i(\mathbf{x}_i)\cdot(\mathbf{r}-\mathbf{x}_i) - \frac{1}{2}R_i^3\mathbf{v}_i(\mathbf{x}_i)\cdot\nabla_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\\ &\qquad + (\text{higher order harmonics})\Bigg)\Bigg|^2 ds\\ =&-\mu\sum_{i=1}^N\int_{S_i}\frac{\partial}{\partial n}\Bigg| - R_i^2\dot{R}_i\nabla_r\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right) + \frac{1}{2}R_i^3\nabla_{\mathbf{r}}^2\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\cdot(\mathbf{U}_i - \mathbf{v}_i)\\ &\qquad + \mathbf{v}_i + (\text{higher order harmonics})\Bigg|^2 ds.\end{aligned}$$

Since spherical harmonics of different order are orthogonal to each other, we calculate each term in the above equation separately. For the monopole term, we have

$$\begin{aligned}\int_{S_i} \frac{\partial}{\partial n}\left|\nabla\left(-\frac{R_i^2\dot{R}_i}{|\mathbf{r}-\mathbf{x}_i|}\right)\right|^2 ds &= \int_{S_i}\frac{\partial}{\partial n}\left[\frac{R_i^4\dot{R}_i^2}{|\mathbf{r}-\mathbf{x}_i|^4}\right] ds\\ &= \int_{S_i}\frac{-4\dot{R}_i^2}{|\mathbf{r}-\mathbf{x}_i|} ds\\ &= -16\pi R_i\dot{R}_i^2.\end{aligned}$$

A direct calculation from Mathematica shows us that

$$\begin{aligned}&\int_{S_i}\frac{\partial}{\partial n}\left(\frac{\partial^2}{\partial x^2}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\right)^2 ds = -\frac{96\pi}{5R_i^5},\\ &\int_{S_i}\frac{\partial}{\partial n}\left(\frac{\partial^2}{\partial x^2}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\frac{\partial^2}{\partial y^2}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\right) ds = \frac{48\pi}{5R_i^5},\\ &\int_{S_i}\frac{\partial}{\partial n}\left(\frac{\partial^2}{\partial x^2}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\right) ds = 0,\\ &\int_{S_i}\frac{\partial}{\partial n}\left(\frac{\partial^2}{\partial x^2}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\frac{\partial^2}{\partial y\partial z}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\right) ds = 0,\\ &\int_{S_i}\frac{\partial}{\partial n}\left(\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\right) = -\frac{72\pi}{5R_i^5},\\ &\int_{S_i}\frac{\partial}{\partial n}\left(\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\frac{\partial^2}{\partial x\partial z}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\right) = 0.\end{aligned}$$

With these results, we can calculate dipole terms in energy dissipation and find

$$\begin{aligned}&\int_{S_i}\frac{\partial}{\partial n}\left|\nabla\left(\frac{1}{2}R_i^3\nabla_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{x}_i|}\right)\cdot(\mathbf{U}_i-\mathbf{v}_i(\mathbf{x}_i))\right)\right|^2 ds\\ &= -\frac{1}{4}R_i^6\left(\frac{96\pi}{5R_i^5}+\frac{72\pi}{5R_i^5}+\frac{72\pi}{5R_i^5}\right)|\mathbf{U}_i-\mathbf{v}_i(\mathbf{x}_i)|^2\\ &= -12\pi R_i|\mathbf{U}_i-\mathbf{v}_i(\mathbf{x}_i)|^2.\end{aligned}$$

Hence one finds

$$\begin{aligned}\mathcal{D} = \sum_{i=1}^N \mu\pi R_i\left(16\dot{R}_i^2+12|\mathbf{U}_i-\mathbf{v}_i(\mathbf{x}_i)|^2\right)\\ +\text{(terms caused by higher order harmonics).}\end{aligned}$$

Drag force. We assume there is no radial oscillation and all bubbles have same radius R. Then

$$\begin{aligned}\mathbf{F}_i &= -\frac{1}{2}\frac{\partial\mathcal{D}}{\partial\mathbf{U}_i} = -6\mu\pi R\frac{\partial}{\partial\mathbf{U}_i}\sum_{j=1}^N|\mathbf{U}_j-\mathbf{v}_j|^2\\ &= -6\mu\pi R\frac{\partial}{\partial\mathbf{U}_i}\sum_{j=1}^N(|\mathbf{U}_j|^2-2\mathbf{U}_j\cdot\mathbf{v}_j(\mathbf{x}_j)+|\mathbf{v}_j(\mathbf{x}_j)|^2)\\ &= -12\mu\pi R\left(\mathbf{U}_i-\frac{\partial}{\partial\mathbf{U}_i}\sum_{j=1}^N\mathbf{U}_j\cdot\mathbf{v}_j(\mathbf{x}_j)+\frac{1}{2}\frac{\partial}{\partial\mathbf{U}_i}\sum_{j=1}^N|\mathbf{v}_j(\mathbf{x}_j)|^2\right).\end{aligned}\tag{117}$$

From the expression for the kinetic energy K in (20), we have, when radial oscillations are absent,

$$\sum_{j=1}^{N} \mathbf{U}_j \cdot \mathbf{v}_j(\mathbf{x}_j) = -\frac{K}{\pi\rho_\ell R^3} + \frac{1}{3}\sum_{j=1}^{N} |\mathbf{U}_j|^2.$$

Hence one finds

$$\frac{\partial}{\partial \mathbf{U}_i}\sum_{j=1}^{N} \mathbf{U}_j \cdot \mathbf{v}_j(\mathbf{x}_j) = -\frac{1}{\pi\rho_\ell R^3}\frac{\partial K}{\partial \mathbf{U}_i} + \frac{2}{3}\mathbf{U}_i.$$

Using (23), we have

$$\frac{\partial}{\partial \mathbf{U}_i}\sum_{j=1}^{N} \mathbf{U}_j \cdot \mathbf{v}_j(\mathbf{x}_j) = 2\mathbf{v}_i(\mathbf{x}_i). \tag{118}$$

Unfortunately we cannot calculate exactly

$$\frac{\partial}{\partial \mathbf{U}_i}\sum_{j=1}^{N} |\mathbf{v}_j(\mathbf{x}_j)|^2,$$

and we will use (14) to provide an approximate calculation. Using the first term in (14), we have

$$\phi \approx \sum_{j=1}^{N} \phi_i,$$

and it follows that

$$\psi_j \approx \sum_{k\neq j, k=1}^{N} \phi_k.$$

With the expression for ϕ_j in (13), we have

$$\frac{\partial \mathbf{v}_j(\mathbf{x}_j)}{\partial \mathbf{U}_i} = \frac{\partial \nabla\psi_j(\mathbf{x}_j)}{\partial \mathbf{U}_i} \approx \begin{cases} \frac{1}{2}R_i^3\nabla^2\left(\frac{1}{|\mathbf{x}_i-\mathbf{x}_j|}\right), & i \neq j, \\ 0, & i = j. \end{cases}$$

Therefore we find (with $R_i = R$)

$$\frac{\partial}{\partial \mathbf{U}_i}\sum_{j=1}^{N} |\mathbf{v}_j(\mathbf{x}_j)|^2 = 2\sum_{j=1}^{N}\frac{\partial \mathbf{v}_j(\mathbf{x}_j)}{\partial \mathbf{U}_i}\cdot \mathbf{v}_j(\mathbf{x}_j) = \sum_{j=1, j\neq i}^{N} R^3\nabla^2\left(\frac{1}{|\mathbf{x}_i-\mathbf{x}_j|}\right)\cdot \mathbf{v}_j(\mathbf{x}_j). \tag{119}$$

Using (117), (118), and (119), we obtain

$$\mathbf{F}_i \approx -12\pi\mu R(\mathbf{U}_i - 2\mathbf{v}_i(\mathbf{x}_i) - \mathbf{w}_i(\mathbf{x}_i)),$$

where

$$\mathbf{w}_i(\mathbf{r}) = -\sum_{j\neq i}\frac{1}{2}R^3\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{x}_j|}\right)\cdot \mathbf{v}_j(\mathbf{x}_j).$$

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