CHAPTER 2 EXERCISES

1. (a) Step 1: Ask the question.

Variables: x = Blue whale population (whales)
y = Fin whale population (whales)
B = Blue whale population growth rate (whales / year)
F = Fin whale population growth rate (whales / year)
T = Total population growth rate (whales / year)

Assumptions: B = 0.05 x (1 - x / 150000) - 10^(-8) x y
F = 0.08 y (1 - y / 400000) - 10^(-8) x y
T = B + F
x >= 0
y >= 0
Objective: Maximize T.

Step 2: Select the modeling approach.

We will model this problem as a multivariable unconstrained optimization problem. See text p. 23.

Step 3: Formulate the model.

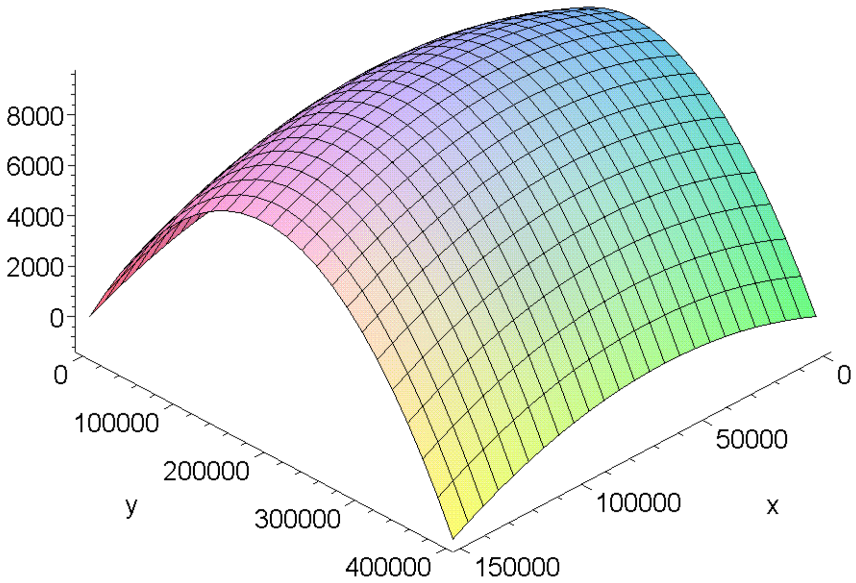
Let z = T, and write
z = f(x, y) = 0.05 x (1 - x / 150000) - 10^(-8) x y
+ 0.08 y (1 - y / 400000) - 10^(-8) x y
Our goal is to maximize f(x, y) over the set of all (x, y) for which x >= 0 and y >= 0.

Step 4: Solve the model.

```
> z:=0.05*x*(1 - x / 150000) - 10^(-8)* x* y + 0.08* y* (1 - y /
  400000) - 10^(-8)* x*y;
```

$$z := .05\,x\left(1-\frac{1}{150000}\,x\right)-\frac{1}{50000000}\,x\,y+.08\,y\left(1-\frac{1}{400000}\,y\right)$$

```
> plot3d(z, x=0..150000,y=0..400000,axes=framed,tickmarks=[4,5,5]);
```



```
> dzdx:=diff(z,x);
```

$$dzdx := .05 - .6666666666\ 10^{-6}\ x - \frac{1}{50000000}\ y$$

```
> dzdy:=diff(z,y);
```

$$dzdy := -\frac{1}{50000000}\ x + .08 - .4000000000\ 10^{-6}\ y$$

```
> solve({dzdx=0,dzdy=0},{x,y});
```

$$\{y = 196544.8172,\ x = 69103.65549\}$$

```
> assign(%);
> z;
```

$$9589.384075$$

The graph indicates that the maximum occurs at the unique point where the gradient vector is zero. This point is x = 69103, y = 196545, z = 9589.

Step 5: Answer the question.

A population of 196,545 Fin whales and 69,103 Blue whales will result in 9,589 whale births per year, which is the maximum.

(b) Let r1 denote the growth rate for Blue whales, where currently we assume r1 = 0.05. Generalizing the calculations of part (a) we now have

```
> restart:
> z:=r1*x*(1 - x / 150000) - 10^(-8)* x* y + 0.08* y* (1 - y /
  400000) - 10^(-8)* x*y;
```

$$z := r1\, x \left(1 - \frac{1}{150000} x\right) - \frac{1}{50000000} x\, y + .08\, y \left(1 - \frac{1}{400000} y\right)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := r1 \left(1 - \frac{1}{150000} x\right) - \frac{1}{150000} r1\, x - \frac{1}{50000000} y$$

$$dzdy := -\frac{1}{50000000} x + .08 - .4000000000\ 10^{-6}\, y$$

```
> s:=solve({dzdx=0,dzdy=0},{x,y});
> assign(s);
```

$$s := \left\{y = .7850000000\ 10^{10} \frac{r1}{40000.\ r1 - 3.},\ x = .12000000\ 10^{8} \frac{250.\ r1 - 1.}{40000.\ r1 - 3.}\right\}$$

```
> dxdr1:=diff(x,r1);
> dydr1:=diff(y,r1);
> assign(r1=0.05);
```

$$dxdr1 := .3000000000\ 10^{10} \frac{1}{40000.\ r1 - 3.} - \frac{.4800000000\ 10^{12}\, (250.\ r1 - 1.)}{(40000.\ r1 - 3.)^2}$$

$$dydr1 := -.3140000000\ 10^{15} \frac{r1}{(40000.\ r1 - 3.)^2} + \frac{.7850000000\ 10^{10}}{40000.\ r1 - 3.}$$

```
> sxr1:=dxdr1*(r1/x);
> syr1:=dydr1*(r1/y);
```

$$sxr1 := .08545426807$$

$$syr1 := -.001502253044$$

Then S(x, r_1) = +0.085 and S(y, r_1) = -0.0015 and so both optimal population levels are quite insensitive to the intrinsic growth rate for Blue whales. Now let r2 denote the intrinsic growth rate for Fin whales, where currently we assume r1 = 0.08. Calculate that

```
> restart:
> z:=0.05*x*(1 - x / 150000) - 10^(-8)* x* y + r2* y* (1 - y /
  400000) - 10^(-8)* x*y;
```

$$z := .05\, x \left(1 - \frac{1}{150000} x\right) - \frac{1}{50000000} x\, y + r2\, y \left(1 - \frac{1}{400000} y\right)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := .05 - .6666666666\ 10^{-6}\ x - \frac{1}{50000000}\, y$$

$$dzdy := -\frac{1}{50000000}\, x + r2\left(1 - \frac{1}{400000}\, y\right) - \frac{1}{400000}\, r2\ y$$

```
> s:=solve({dzdx=0,dzdy=0},{x,y});
> assign(s);
```

$$s := \{y = 200000.\ \frac{-.5000000\ 10^{7} + .3333333333\ 10^{10}\ r2}{.3333333333\ 10^{10}\ r2 - 400000.},$$

$$x = .2300000000\ 10^{15}\ \frac{r2}{.3333333333\ 10^{10}\ r2 - 400000.}\}$$

```
> dxdr2:=diff(x,r2):
> dydr2:=diff(y,r2):
> assign(r2=0.08);
> sxr2:=dxdr2*(r2/x);
> syr2:=dydr2*(r2/y);
```

$$sxr2 := \text{-}.001502254074$$

$$syr2 := .01760602701$$

Then S(x, r_2) = -0.0015 and S(y, r_2) = +0.0176 and so both optimal population levels are also quite insensitive to the intrinsic growth rate for Fin whales. Understandably the optimal populations are more sensitive to their own growth rate than to that of the other population.

(c) Now let K1 be the carrying capacity for Blue whales, where currently K1 = 150000.

```
> unassign('x','y');
> z:=0.05*x*(1 - x / K1) - 10^(-8)* x* y + 0.08* y* (1 - y / 400000)
  - 10^(-8)* x*y;
```

$$z := .05\ x\left(1 - \frac{x}{K1}\right) - \frac{1}{50000000}\, x\, y + .08\ y\left(1 - \frac{1}{400000}\, y\right)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := .05 - \frac{.10\ x}{K1} - \frac{1}{50000000}\, y$$

$$dzdy := -\frac{1}{50000000}\, x + .08 - .4000000000\ 10^{-6}\ y$$

```
> s:=solve({dzdx=0,dzdy=0},{x,y});
> assign(s);
```

$$s := \{y = .2500000\ 10^{7}\ \frac{K1 - .8000000\ 10^{7}}{-.100000000\ 10^{9} + K1},\ x = -.46000000\ 10^{8}\ \frac{K1}{-.100000000\ 10^{9} + K1}\}$$

```
> dxdK1:=diff(x,K1):
> dydK1:=diff(y,K1):
> assign(K1=150000);
> sxK1:=dxdK1*(K1/x);
> syK1:=dydK1*(K1/y);
```

$$sxK1 := 1.001502253$$

$$syK1 := -.01760602687$$

Then S(x, K_1) = +1.0015 and S(y, K_1) = -0.0176 so that if the carrying capacity for Blue whales increases by 10% then the optimal population for Blue whales increases by about 10% and the optimal population for Fin whales stays about the same. Now compute that

```
> restart:
> z:=0.05*x*(1 - x / 150000) - 10^(-8)* x* y + 0.08* y* (1 - y / K2)
  - 10^(-8)* x*y;
```

$$z := .05\,x\left(1-\frac{1}{150000}\,x\right)-\frac{1}{50000000}\,x\,y+.08\,y\left(1-\frac{y}{K2}\right)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := .05-.6666666666\;10^{-6}\,x-\frac{1}{50000000}\,y$$

$$dzdy := -\frac{1}{50000000}\,x+.08-\frac{.16\,y}{K2}$$

```
> s:=solve({dzdx=0,dzdy=0},{x,y});
> assign(s);
```

$$s := \{y=-.3270833333\;10^{10}\,\frac{K2}{25.\,K2-.6666666666\;10^{10}},$$

$$x=.100000000\;10^{9}\,\frac{K2-.5000000\;10^{7}}{25.\,K2-.6666666666\;10^{10}}\}$$

```
> dxdK2:=diff(x,K2):
> dydK2:=diff(y,K2):
> assign(K2=400000);
> sxK2:=dxdK2*(K2/x);
> syK2:=dydK2*(K2/y);
```

$$sxK2 := -.08545426835$$

$$syK2 := 1.001502253$$

Then S(x, K_2) = -0.085 and S(y, K_2) = +1.0015 so that if the carrying capacity for Fin whales increases by 10% then the optimal population for Fin whales increases by about 10% and the optimal population for Blue whales stays about the same.

(d) Now we let a = alpha be the competition parameter, and we calculate

```
> unassign('x','y');
> z:=0.05*x*(1 - x / 150000) - a* x* y + 0.08* y* (1 - y / K2) - a*
  x*y;
```

$$z := .05\,x\left(1-\frac{1}{150000}\,x\right)-2\,a\,x\,y+.08\,y\left(1-\frac{1}{400000}\,y\right)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := .05-.6666666666\;10^{-6}\,x-2\,a\,y$$

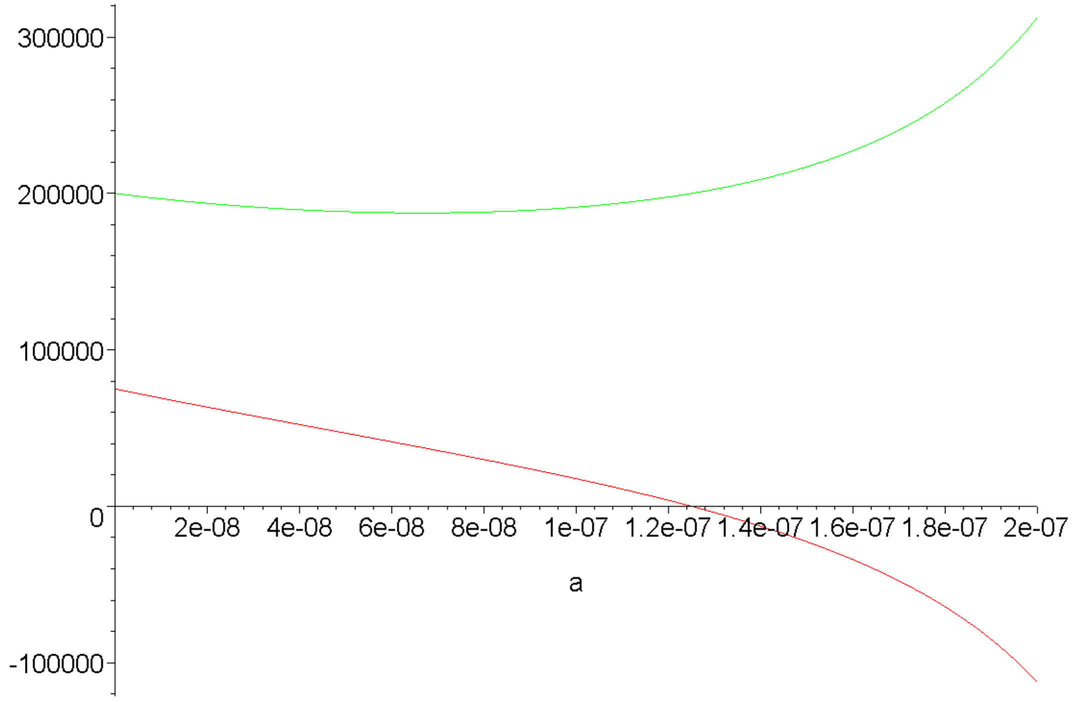
$$dzdy := -2\,a\,x+.08-.4000000000\;10^{-6}\,y$$

```
> s:=solve({dzdx=0,dzdy=0},{x,y});
> assign(s);
```

$$s := \{x=.2500000000\;10^{15}\,\frac{-1.+.8000000\;10^{7}\,a}{-.3333333333\;10^{10}+.5000000000\;10^{23}\,a^{2}},$$

$$y=200000.\,\frac{.6250000000\;10^{16}\,a-.3333333333\;10^{10}}{-.3333333333\;10^{10}+.5000000000\;10^{23}\,a^{2}}\}$$

```
> plot({x,y},a=0..2*10^(-7));
> solve(x=0);
```



$$.1250000000\;10^{-6}$$

so that when alpha_1 = alpha_2 =a > .000000125 it is optimal to extinct the Blue whales.

2. (a) Step 1: Ask the question.

Variables: x = Blue whale population (whales)
y = Fin whale population (whales)
B = Blue whale population growth rate (whales / year)
F = Fin whale population growth rate (whales / year)
P = Total population (whales)

Assumptions: $B = 0.05\, x\, (1 - x / 150000) - 10^{(-8)}\, x\, y$
$F = 0.08\, y\, (1 - y / 400000) - 10^{(-8)}\, x\, y$
$P = x + y$
$x >= 0, \; y >= 0$
$B >= 0, F >= 0$

Objective: Maximize P.

Step 2: Select the modeling approach.

We will model this problem as a multivariable constrained optimization problem which will be solved using the method of Lagrange multipliers. See text p. 33.

Step 3: Formulate the model.

Let $z = P$, and write

$$z = f(x, y) = x + y$$

we wish to maximize z over the set of $x >= 0$ and $y >= 0$ which satisfy the additional constraints

$$0 <= g1(x, y) = 0.05\, x\, (1 - x / 150000) - 10^{(-8)}\, x\, y$$

$$0 <= g1(x, y) = 0.08\, y\, (1 - y / 400000) - 10^{(-8)}\, x\, y$$

Step 4: Solve the model.

First we consider the unconstrained model. Since $z = x + y$ is linear there is no local extrema in the interior of the feasible region. In fact the gradient vector of $f(x, y)$ is (1, 1) everywhere. Then the maximum must occur on the boundary. The feasible region is bounded by the lines $x = 0$, $y = 0$, $g1/ x = 0$, $g2 / y = 0$. The gradient vector (1, 1) will never be perpendicular to the lines $x = 0$ or $y = 0$ so there are no local max or min on these segments of the boundary. Next we check the other parts of the boundary.

```
> unassign('x','y');
> g1:=0.05*x*(1 - x / 150000) - 10^(-8)* x* y;
> g2:=0.08*y*(1 - y / 400000) - 10^(-8)* x* y;
```

$$g1 := .05\,x\left(1-\frac{1}{150000}\,x\right)-\frac{1}{100000000}\,x\,y$$

$$g2 := .08\,y\left(1-\frac{1}{400000}\,y\right)-\frac{1}{100000000}\,x\,y$$

```
> dg1dx:=diff(g1,x);
> dg1dy:=diff(g1,y);
> dg2dx:=diff(g2,x);
> dg2dy:=diff(g2,y);
```

$$dg1dx := .05-.6666666666\;10^{-6}\,x-\frac{1}{100000000}\,y$$

$$dg1dy := -\frac{1}{100000000}\,x$$

$$dg2dx := -\frac{1}{100000000}\,y$$

$$dg2dy := .08-.4000000000\;10^{-6}\,y-\frac{1}{100000000}\,x$$

```
> solve({dg1dx=1,dg1dy=1},{x,y});
> solve({dg2dx=1,dg2dy=1},{x,y});
```

$$\{x=-.100000000\;10^{9},\; y=.6571666666\;10^{10}\}$$

$$\{y=-.100000000\;10^{9},\; x=.3908000000\;10^{10}\}$$

Here also there are no local max or min since the Lagrange multiplier equations have no solution in the feasible region. Then the max (and the min) must occur at one of the corners. At (0,0) we have the minimum z = 0. Now we will check the other corner points.

```
> solve({y=0,g1/x=0});
```

$$\{y=0.,\; x=150000.\}$$

```
> solve({x=0,g2/y=0});
```

$$\{x=0.,\; y=400000.\}$$

```
> solve({g1/x=0,g2/y=0});
```

$$\{y=393089.6345,\; x=138207.3110\}$$

```
> assign(%);x+y;
```

$$531296.9455$$

Then the maximum occurs at the only corner point at which both x and y are positive. At this corner point (138207, 393090) we have the maximum value z = 531297.

Step 5: Answer the question.

A population of 393,090 Fin whales and 138,207 Blue whales will result in a total of 531,297 whales, which is the maximum. At these population levels, both growth rates are zero, and so both populations will remain the same as time goes on. This is the largest population that the ecosystem can support. It is slightly less than the theoretical maximum of 550,000 obtained by summing the carrying capacity for both species, because of the effects of competition.

(b) Let r1 denote the growth rate for Blue whales, where currently we assume r1 = 0.05. Generalizing the calculations of part (a) we now have

```
> unassign('x','y','r1');
> g1:=r1*x*(1 - x / 150000) - 10^(-8)* x* y;
> g2:=0.08*y*(1 - y / 400000) - 10^(-8)* x* y;
```

$$g1 := r1\, x \left(1 - \frac{1}{150000}\, x\right) - \frac{1}{100000000}\, x\, y$$

$$g2 := .08\, y \left(1 - \frac{1}{400000}\, y\right) - \frac{1}{100000000}\, x\, y$$

```
> solve({g1/x=0,g2/y=0},{x,y});
```

$$\{x = .24000000\ 10^{8}\, \frac{250.\, r1 - 1.}{40000.\, r1 - 3.},\ y = .1570000000\ 10^{11}\, \frac{r1}{40000.\, r1 - 3.}\}$$

```
> assign(%);
> dxdr1:=diff(x,r1);
> dydr1:=diff(y,r1);
```

$$dxdr1 := .6000000000\ 10^{10}\, \frac{1}{40000.\, r1 - 3.} - \frac{.9600000000\ 10^{12}\, (250.\, r1 - 1.)}{(40000.\, r1 - 3.)^2}$$

$$dydr1 := -.6280000000\ 10^{15}\, \frac{r1}{(40000.\, r1 - 3.)^2} + \frac{.1570000000\ 10^{11}}{40000.\, r1 - 3.}$$

```
> assign(r1=0.05):
> sxr1:=dxdr1*(r1/x);
> syr1:=dydr1*(r1/y);
```

$$sxr1 := .08545426841$$

$$syr1 := -.001502253044$$

Then S(x, r_1) = +0.085 and S(y, R_1) = -0.0015 and so both optimal population levels are quite insensitive to the intrinsic growth rate for Blue whales. Now let r2 denote the intrinsic growth rate for Fin whales, where currently we assume r1 = 0.08. Calculate that

```
> unassign('x','y','r2');
> g1:=0.05*x*(1 - x / 150000) - 10^(-8)* x* y;
> g2:=r2*y*(1 - y / 400000) - 10^(-8)* x* y;
```

$$g1 := .05\, x \left(1 - \frac{1}{150000}\, x\right) - \frac{1}{100000000}\, x\, y$$

$$g2 := r2\, y \left(1 - \frac{1}{400000}\, y\right) - \frac{1}{100000000}\, x\, y$$

```
> solve({g1/x=0,g2/y=0},{x,y});
> assign(%);
```

$$\{x = .3450000000\ 10^{10} \frac{r2}{-3. + 25000.\ r2},\ y = .5000000\ 10^{7} \frac{-3. + 2000.\ r2}{-3. + 25000.\ r2}\}$$

```
> dxdr2:=diff(x,r2):
> dydr2:=diff(y,r2):
> assign(r2=0.08);
> sxr2:=dxdr2*(r2/x);
> syr2:=dydr2*(r2/y);
```

$$sxr2 := -.001502253379$$

$$syr2 := .01760602680$$

Then S(x, r_2) = -0.0015 and S(y, r_2) = +0.0176 and so both optimal population levels are also quite insensitive to the intrinsic growth rate for Fin whales. Understandably the optimal populations are more sensitive to their own growth rate than to that of the other population.

(c) Now let K1 be the carrying capacity for Blue whales, where currently K1 = 150000.

```
> unassign('x','y','K1');
> g1:=0.05*x*(1 - x / K1) - 10^(-8)* x* y;
> g2:=0.08*y*(1 - y / 400000) - 10^(-8)* x* y;
```

$$g1 := .05\, x \left(1 - \frac{x}{K1}\right) - \frac{1}{100000000}\, x\, y$$

$$g2 := .08\, y \left(1 - \frac{1}{400000}\, y\right) - \frac{1}{100000000}\, x\, y$$

```
> solve({g1/x=0,g2/y=0},{x,y});
> assign(%);
```

$$\{y = .5000000\ 10^{7} \frac{K1 - .8000000\ 10^{7}}{-.100000000\ 10^{9} + K1},\ x = -.92000000\ 10^{8} \frac{K1}{-.100000000\ 10^{9} + K1}\}$$

```
> dxdK1:=diff(x,K1):
> dydK1:=diff(y,K1):
> assign(K1=150000);
> sxK1:=dxdK1*(K1/x);
> syK1:=dydK1*(K1/y);
```

$$sxK1 := 1.001502253$$

$$syK1 := -.01760602687$$

Then S(x, K_1) = +1.00015 and S(y, K_1) = -0.0176 so that if the carrying capacity for Blue whales increases by 10% then the optimal population for Blue whales increases by about 10% and the optimal population for Fin whales stays about the same. Now compute that

```
> unassign('x','y','K2');
```

```
> g1:=0.05*x*(1 - x / 150000) - 10^(-8)* x* y;
> g2:=0.08*y*(1 - y / K2) - 10^(-8)* x* y;
```

$$g1 := .05\,x\left(1-\frac{1}{150000}\,x\right)-\frac{1}{100000000}\,x\,y$$

$$g2 := .08\,y\left(1-\frac{y}{K2}\right)-\frac{1}{100000000}\,x\,y$$

```
> solve({g1/x=0,g2/y=0},{x,y});
> assign(%);
```

$$\left\{x=.24000000\;10^{8}\,\frac{K2-.5000000\;10^{7}}{3.\,K2-.800000000\;10^{9}},\;y=-.785000000\;10^{9}\,\frac{K2}{3.\,K2-.800000000\;10^{9}}\right\}$$

```
> dxdK2:=diff(x,K2):
> dydK2:=diff(y,K2):
> assign(K2=400000);
> sxK2:=dxdK2*(K2/x);
> syK2:=dydK2*(K2/y);
```

$$sxK2 := -.08545426833$$

$$syK2 := 1.001502254$$

Then S(x, K_2) = -0.085 and S(y, K_2) = +1.0015 so that if the carrying capacity for Fin whales increases by 10% then the optimal population for Fin whales increases by about 10% and the optimal population for Blue whales stays about the same.

(d) Now we let a = alpha be the competition parameter, and we calculate

```
> unassign('x','y','a');
> g1:=0.05*x*(1 - x / 150000) - a* x* y;
> g2:=0.08*y*(1 - y / 400000) - a* x* y;
```

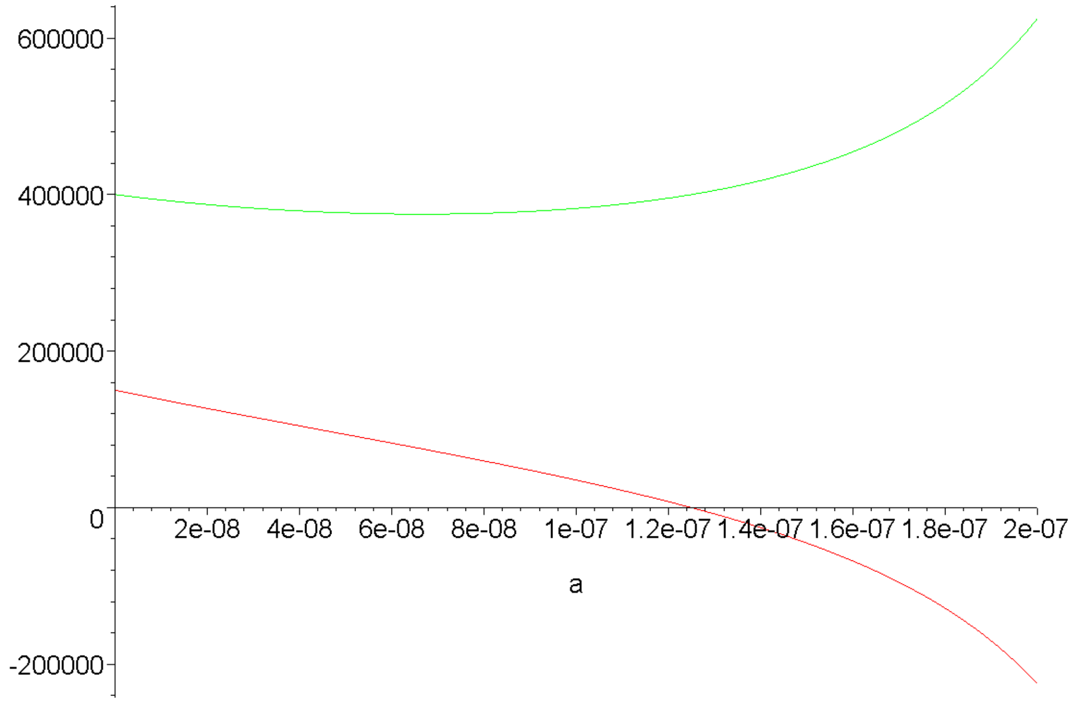
$$g1 := .05\,x\left(1-\frac{1}{150000}\,x\right)-a\,x\,y$$

$$g2 := .08\,y\left(1-\frac{1}{400000}\,y\right)-a\,x\,y$$

```
> solve({g1/x=0,g2/y=0},{x,y});
> assign(%);
```

$$\left\{x=150000.\,\frac{-1.+.8000000\;10^{7}\,a}{-1.+.1500000000\;10^{14}\,a^{2}},\;y=400000.\,\frac{-1.+.1875000\;10^{7}\,a}{-1.+.1500000000\;10^{14}\,a^{2}}\right\}$$

```
> plot({x,y},a=0..2*10^(-7));
> solve(x=0);
```



$.1250000000 \; 10^{-6}$

so that when alpha_1 = alpha_2 =a > .000000125 it is optimal to extinct the Blue whales. If you also solved problem 1, then you saw a marked similarity in answers. This is because the population levels which maximize growth rate are (apparently) always exactly half of the levels which maximize total population.

3. (a) Step 1: Ask the question.

Variables:
x = Blue whale population (whales)
y = Fin whale population (whales)
B = Blue whale harvest rate (whales / year)
F = Fin whale harvest rate (whales / year)
R = Total revenue ($1000 / year)

Assumptions:
B = 0.05 x (1 - x / 150000) - 10^(-8) x y
F = 0.08 y (1 - y / 400000) - 10^(-8) x y
R = 12 B + 6 F
x >= 0
y >= 0

Objective: Maximize R.

Step 2: Select the modeling approach.

We will model this problem as a multivariable unconstrained optimization problem. See text p. 23.

Step 3: Formulate the model.

Let z = R, and write

z = f(x, y) = 12 (0.05 x (1 - x / 150000) - 10^(-8) x y)
+ 6 (0.08 y (1 - y / 400000) - 10^(-8) x y)

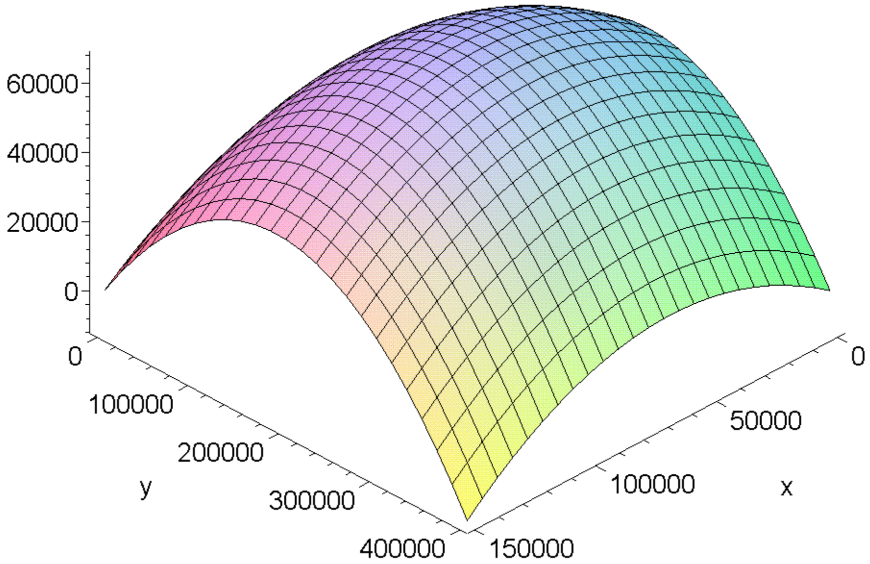
Our goal is to maximize f(x, y) over the set of all (x, y) for which x >= 0 and y >= 0.

Step 4: Solve the model.

```
> restart: z:=12*(0.05*x*(1 - x / 150000)-10^(-8)*x*y)
  +6*(0.08*y*(1 - y / 400000)-10^(-8)* x*y);
```

$$z := .60\, x \left(1 - \frac{1}{150000} x\right) - \frac{9}{50000000} x\, y + .48\, y \left(1 - \frac{1}{400000} y\right)$$

```
> plot3d(z, x=0..150000,y=0..400000,axes=framed,tickmarks=[4,5,5]);
```



```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := .60 - .8000000000\; 10^{-5}\, x - \frac{9}{50000000}\, y$$

$$dzdy := -\frac{9}{50000000}\,x + .48 - .2400000000\;10^{-5}\,y$$

```
> solve({dzdx=0,dzdy=0},{x,y});
> assign(%);
```

$$\{y = 194703.5623,\; x = 70619.16985\}$$

```
> z;
```

$$67914.60589$$

The graph indicates that the maximum occurs at the unique point where the gradient vector is zero. This point is x = 70619, y = 194703, z = 67915.

Step 5: Answer the question.

A population of 194703 Fin whales and 70619 Blue whales will result in the maximum revenue. At these population levels, the annual harvest is worth around $68 million.

(b,c) Let r1 denote the growth rate for Blue whales, where currently we assume r1 = 0.05. Generalizing the calculations of part (a) we now have

```
> unassign('x','y','r1');
> z:=12*(r1*x*(1 - x / 150000)-10^(-8)*x*y)+6*(0.08*y*(1 - y /
  400000)-10^(-8)* x*y);
```

$$z := 12\,r1\,x\left(1 - \frac{1}{150000}\,x\right) - \frac{9}{50000000}\,x\,y + .48\,y\left(1 - \frac{1}{400000}\,y\right)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := 12\,r1\left(1 - \frac{1}{150000}\,x\right) - \frac{1}{12500}\,r1\,x - \frac{9}{50000000}\,y$$

$$dzdy := -\frac{9}{50000000}\,x + .48 - .2400000000\;10^{-5}\,y$$

```
> s:=solve({dzdx=0,dzdy=0},{x,y});
```

$$s := \left\{x = .24000000\;10^{8}\,\frac{-3. + 1000.\,r1}{-27. + 320000.\,r1},\; y = .6220000000\;10^{11}\,\frac{r1}{-27. + 320000.\,r1}\right\}$$

```
> assign(s);
> dxdr1:=diff(x,r1):
> dydr1:=diff(y,r1):
> dzdr1:=diff(z,r1):
> assign(r1=0.05);
> sxr1:=dxdr1*(r1/x);
> syr1:=dydr1*(r1/y);
```

```
> szr1:=dzdr1*(r1/z);
```

$$sxr1 := .06213943480$$

$$syr1 := -.001690352484$$

$$szr1 := .3301680548$$

Then S(x, r_1) = +0.062 and S(y, R_1) = -0.0017 and so both optimal population levels are quite insensitive to the intrinsic growth rate for Blue whales. Also S(z, r_1) = +0.33 so that if the intrinsic growth rate for Blue whales is 10% bigger than expected, then the annual revenue from whaling at the optimal population levels will be 3.3% larger than expected. Now let r2 denote the intrinsic growth rate for Fin whales, where currently we assume r1 = 0.08. Calculate that

```
> unassign('x','y','r2');
> z:=12*(0.05*x*(1 - x / 150000)-10^(-8)*x*y)+6*(r2*y*(1 - y /
  400000)-10^(-8)* x*y);
```

$$z := .60\,x\left(1-\frac{1}{150000}\,x\right)-\frac{9}{50000000}\,x\,y+6\,r2\,y\left(1-\frac{1}{400000}\,y\right)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := .60-.8000000000\;10^{-5}\,x-\frac{9}{50000000}\,y$$

$$dzdy := -\frac{9}{50000000}\,x+6\,r2\left(1-\frac{1}{400000}\,y\right)-\frac{3}{200000}\,r2\,y$$

```
> s:=solve({dzdx=0,dzdy=0},{x,y});
> assign(s);
```

$$s := \{y = .10000000\;10^{8}\,\frac{-9.+4000.\,r2}{-27.+200000.\,r2},\ x = .1410000000\;10^{11}\,\frac{r2}{-27.+200000.\,r2}\}$$

```
> dxdr2:=diff(x,r2):
> dydr2:=diff(y,r2):
> dzdr2:=diff(z,r2):
> assign(r2=0.08);
> sxr2:=dxdr2*(r2/x);
> syr2:=dydr2*(r2/y);
> szr2:=dzdr2*(r2/z);
```

$$sxr2 := -.001690352468$$

$$syr2 := .02724855456$$

$$szr2 := .7062742491$$

Then S(x, r_2) = -0.0017 and S(y, r_2) = +0.0272 and so both optimal population levels are also quite insensitive to the intrinsic growth rate for Fin whales. Understandably the optimal populations are more sensitive to their own growth rate than to that of the other population. Also

S(z, r_2) = +0.71 so that if the intrinsic growth rate of the Fin whale population is 10% greater than expected, then the total annual revenue from whaling will be 7.1% larger than expected.

(d) Now we let a = alpha be the competition parameter, and we calculate

```
> unassign('x','y','a');
> z:=12*(0.05*x*(1 - x / 150000)-a*x*y)+6*(0.08*y*(1 - y /
  400000)-a* x*y);
```

$$z := .60\,x\left(1-\frac{1}{150000}\,x\right)-18\,a\,x\,y+.48\,y\left(1-\frac{1}{400000}\,y\right)$$

```
> dzdx:=diff(z,x);
```

$$dzdx := .60-.8000000000\;10^{-5}\,x-18\,a\,y$$

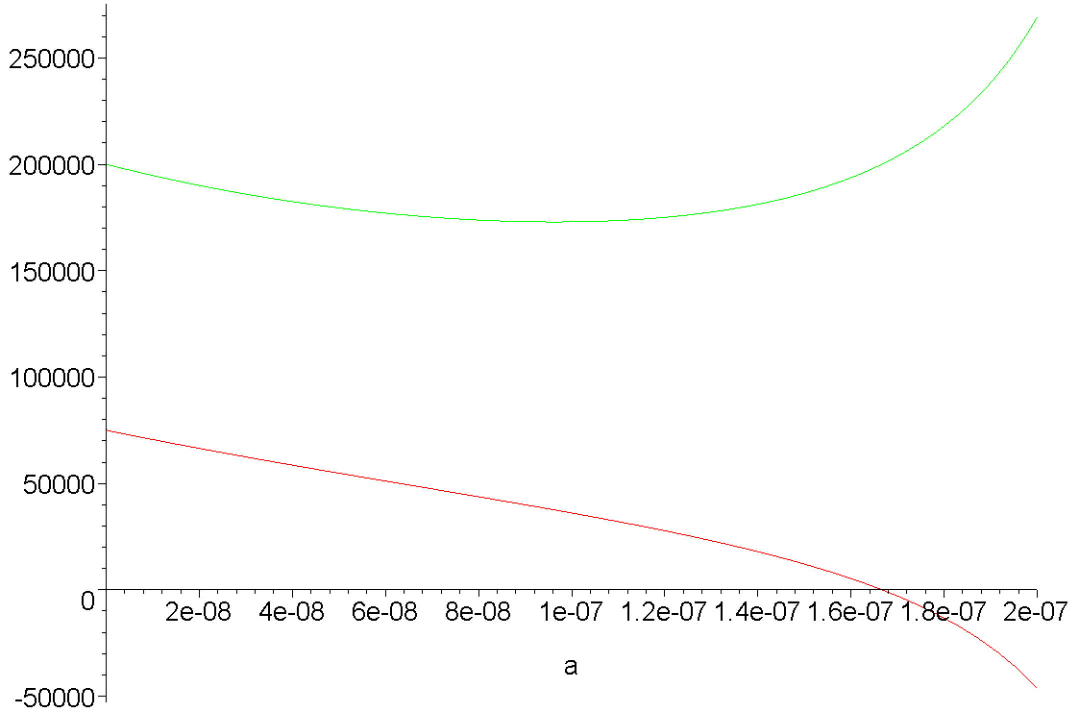
```
> dzdy:=diff(z,y);
```

$$dzdy := -18\,a\,x+.48-.2400000000\;10^{-5}\,y$$

```
> s:=solve({dzdx=0,dzdy=0},{x,y});
```

$$s := \{x=75000.\,\frac{-1.+.6000000\;10^{7}\,a}{.1687500000\;10^{14}\,a^2-1.},\,y=200000.\,\frac{.2812500\;10^{7}\,a-1.}{.1687500000\;10^{14}\,a^2-1.}\}$$

```
> assign(s);
> plot({x,y},a=0..2*10^(-7));
```



```
> solve(x=0);
```

$$.1666666667\;10^{-6}$$

so that when alpha_1 = alpha_2 =a > .000000167 it is optimal to extinct the Blue whales.

4. (a)   Step 1: Ask the question.

Variables: x = Blue whale population (whales)
y = Fin whale population (whales)
B = Blue whale harvest rate (whales / year)
F = Fin whale harvest rate (whales / year)
R = Total revenue ($1000 / year)

Assumptions: B = 0.05 x (1 - x / 150000) - 10^(-8) x y
F = 0.08 y (1 - y / 400000) - 10^(-8) x y
R = 12 B + 6 F
x >= 75,000
y >= 200,000

Objective: Maximize R.

Step 2: Select the modeling approach.

We will model this problem as a multivariable constrained optimization problem, and we will solve the problem using the method of Lagrange multipliers. See text p. 33.

Step 3: Formulate the model.

Let z = R, and write
z = f(x, y) = 12 (0.05 x (1 - x / 150000) - 10^(-8) x y)
+ 6 (0.08 y (1 - y / 400000) - 10^(-8) x y)
Our goal is to maximize f(x, y) over the set of all (x, y) for which x >= 75000 and y >= 200000.

Step 4: Solve the model.

In problem 3 we calculated that the gradient vector is zero when x = 70619, y = 194703. This point is outside the feasible region, so there is no local max or min in the interior of this region. On the line g(x,y) = x = 75000 we use Lagrange multipliers. The gradient of g is (1,0)

```
> restart: z:=12*(0.05*x*(1 - x / 150000)-10^(-8)*x*y)+6*(0.08*y*(1
  - y / 400000)-10^(-8)* x*y);
```

$$z := .60\, x \left(1 - \frac{1}{150000}\, x\right) - \frac{9}{50000000}\, x\, y + .48\, y \left(1 - \frac{1}{400000}\, y\right)$$

```
> dzdx:=diff(z,x);
```

```
> dzdy:=diff(z,y);
```

$$dzdx := .60 - .8000000000\ 10^{-5}\ x - \frac{9}{50000000}\ y$$

$$dzdy := -\frac{9}{50000000}\ x + .48 - .2400000000\ 10^{-5}\ y$$

```
> solve({dzdx=lambda,dzdy=0,x=75000},{x,y,lambda});
```

$$\{y = 194375.,\ x = 75000.,\ \lambda = -.03498750000\}$$

The maximum occurs at y = 194,375 which is not feasible, so the maximum over this segment of the boundary occurs at the endpoint y = 200,000. Next consider the constraint line g(x,y) = y = 200000. Now the gradient of g is (0,1)

```
> z:=12*(0.05*x*(1 - x / 150000)-10^(-8)*x*y)+6*(0.08*y*(1 - y / 
  400000)-10^(-8)* x*y);
```

$$z := .60\ x\left(1 - \frac{1}{150000}\ x\right) - \frac{9}{50000000}\ x\ y + .48\ y\left(1 - \frac{1}{400000}\ y\right)$$

```
> dzdx:=diff(z,x):
> dzdy:=diff(z,y):
> solve({dzdx=0,dzdy=lambda,y=200000},{x,y,lambda});
```

$$\{\lambda = -.01269000000,\ x = 70500.,\ y = 200000.\}$$

```
> assign('x=75000','y=200000');
> z;
```

$$67800.00$$

Once again the maximum occurs at an infeasible point, so the maximum along the boundary segment is at the vertex of the feasible region x = 75,000 and y = 200,000. At this point z = 67,800.

Step 5. The optimal profit will result if both species are kept at 1/2 of the environmental carrying capacity. This will result in an estimated sustained revenue of 67.8 million dollars per year.

(b) Assume x >= c where currently c = 150,000. As long as c > 70619 the optimum point will be at x = c and y = 200,000 and so dx/dc = 1 and dy/dc = 0. Then S(y,c) = 0 and S(x,c) = 1. Similarly if we assume y >= b where currently b = 200,000 then as long as b > 194703 the optimum point will be at the vertex x = 150,000 and y = b so that S(x,b) = 0 and S(y,b) = 1. Then it is easy to calculate that for z = sustained profit (revenue) in $1000's per year we have the sensitivities
S(z,c) = 12 (75000/67800) = +13.27 and S(z,b) = 6 (200000/67800) = +17.70.

(c) the quotas are 1725 Blue whales and 7850 fin whales per year, as shown below. So if we maximize revenue R subject to the harvest constraints B >= 1725 and F >= 7850 we would get the same maximum as before.

```
> B:=0.05*x*(1 - x / 150000)-10^(-8)*x*y;
> F:=0.08*y*(1 - y / 400000)-10^(-8)* x*y;
```

$$B := 1725.00$$

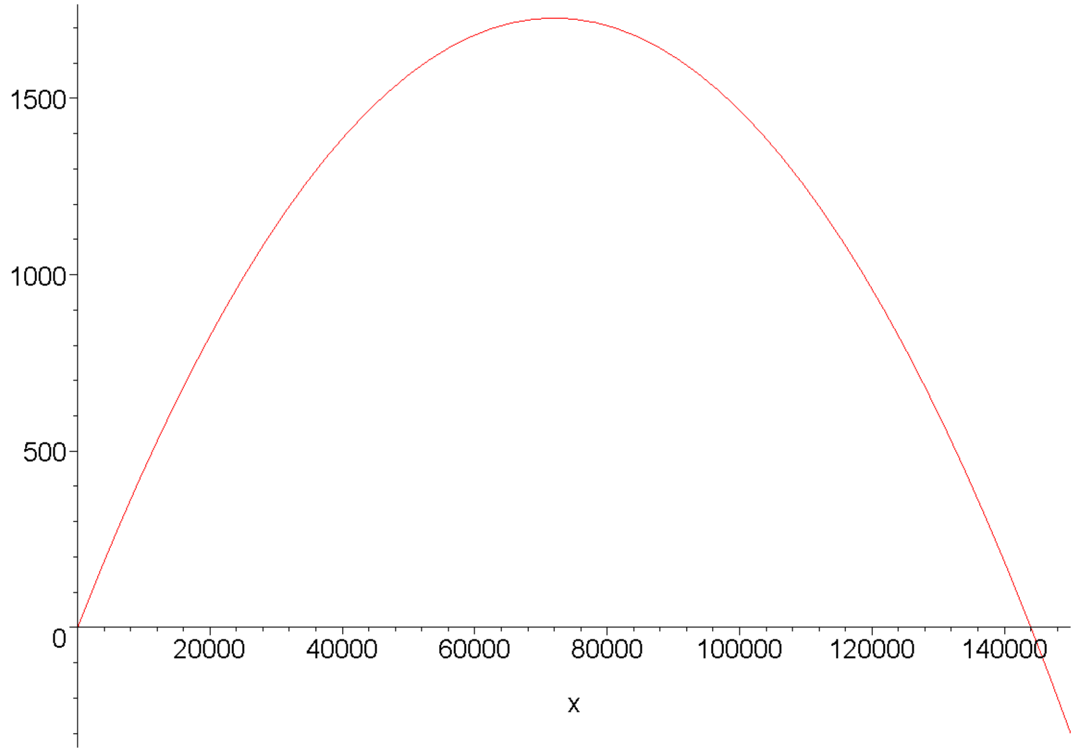
$$F := 7850.00$$

(d) Since the maximum possible growth rates (and hence the maximum possible sustained harvest rates) for the Blue whale and the Fin whale are 1728 and 7851 respectively, the yearly quotas cannot be raised by any significant amount. If the number of whale kills exceeds the current quotas, this will resul in the eventual decline and extinction of one or both species.

```
> unassign('x','y');
> assign('y=200000');
> B:=0.05*x*(1 - x / 150000)-10^(-8)*x*y;
```

$$B := .05\,x\left(1-\frac{1}{150000}\,x\right)-\frac{1}{500}\,x$$

```
> plot(B,x=0..150000);
```



```
> dBdx:=diff(B,x);
```

$$dBdx := .04800000000 - .6666666666\ 10^{-6}\ x$$

```
> solve({dBdx=0},{x});
```

$$\{x = 72000.00001\}$$

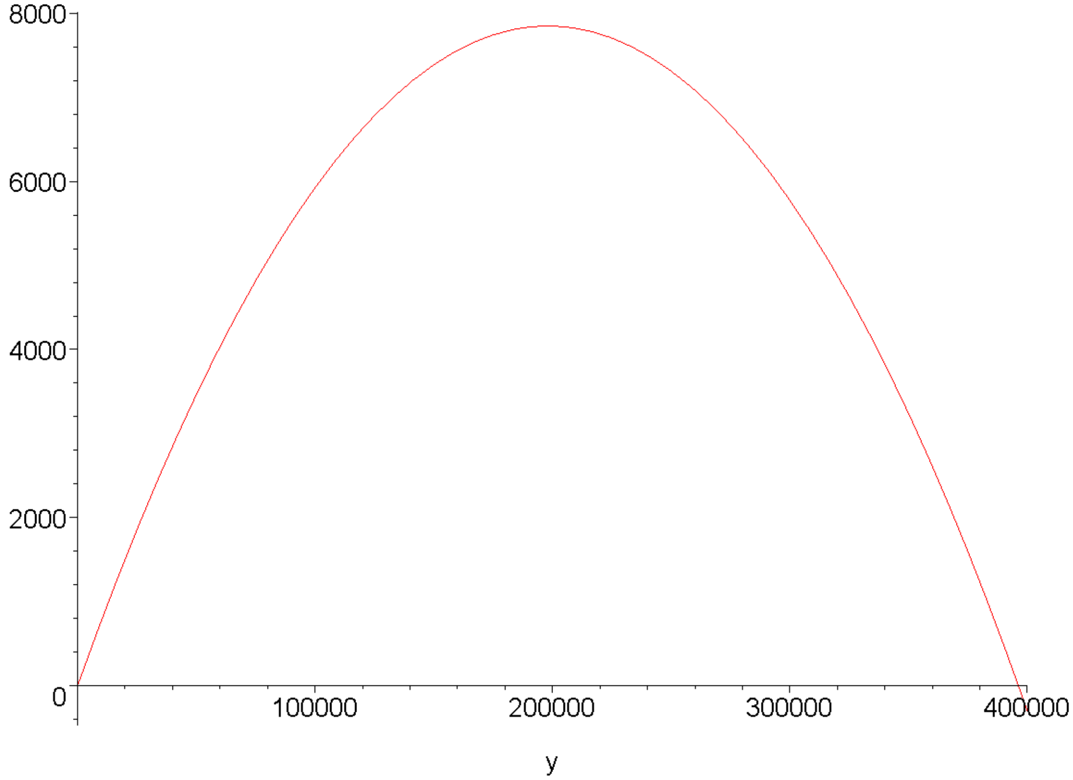
```
> assign(%);
> B;
```

$$1728.000000$$

```
> unassign('x','y');
> assign('x=75000');
> F:=0.08*y*(1 - y / 400000)-10^(-8)* x*y;
```

$$F := .08\,y\left(1 - \frac{1}{400000}\,y\right) - \frac{3}{4000}\,y$$

```
> plot(F,y=0..400000);
```



```
> dFdy:=diff(F,y);
```

$$dFdy := .07925000000 - .4000000000\,10^{-6}\,y$$

```
> solve({dFdy=0},{y});
> assign(%);
```

$$\{y = 198125.\}$$

```
> F;
```

$$7850.703125$$

```
>
```

5. (a) Step 1: Ask the question.

Variables: s = sales of 19 inch sets (sets / year)
t = sales of 21 inch sets (sets / year)
p = price for 19 inch sets ($ / set)
q = price for 21 inch sets ($ / set)
T = tarriff ($ / set)
C = manufacturing cost ($ / year)
R = revenue ($ / year)
P = profit ($ / year)

Assumptions: p = 339 - .01 s - .003 t
q = 399 - .004 s - .01 t
T = 25 (s + t)
R = p s + q t
C = 400,000 + 195 s + 225 t + T
P = R - C
s >= 0
t >= 0

Objective: Maximize P.

Step 2: Select the modeling approach.

We will model this problem as a multivariable unconstrained optimization problem. See text p. 23.

Step 3: Formulate the model.

Let x = s, y = t, and z = P, and write

$$z = f(x, y) = (339 - .01\, x - .003\, y)\, x + (399 - .004\, x - .01\, y)\, y - (400000 + 195\, x + 225\, y + 25\, (x + y))$$

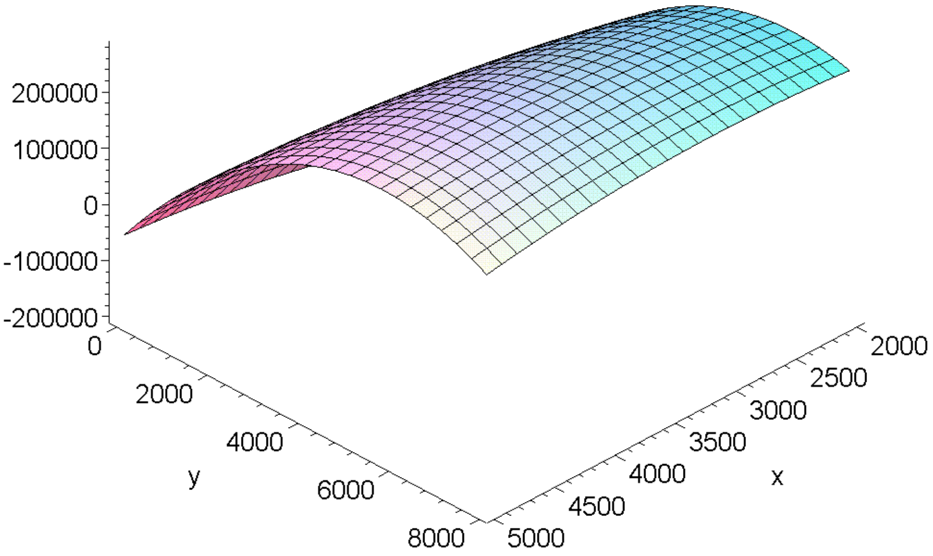
Our goal is to maximize f(x, y) over the set of all (x, y) for which x >= 0 and y >= 0.

Step 4: Solve the model.

```
> unassign('x','y');
> z := (339- .01*x - .003*y)*x +(399 - .004*x - .01*y)*y-(400000 +
  195*x + 225*y + 25*(x + y));
```

$$z := (339 - .01\, x - .003\, y)\, x + (399 - .004\, x - .01\, y)\, y - 400000 - 220\, x - 250\, y$$

```
> plot3d(z, x=2000..5000,y=0..8000,axes=framed);
```



```
> dzdx:=diff(z,x);
                    dzdx := -.02 x + 119. - .007 y
> dzdy:=diff(z,y);
                    dzdy := -.007 x - .02 y + 149.
> solve({dzdx=0,dzdy=0},{x,y});
                 {x = 3809.116809, y = 6116.809117}
> assign(%);
> z;
                              282344.729
```

The graph indicates that the maximum occurs at the unique point where the gradient is zero. We solve to find x = 3809, y = 6117, and z = 282,345 at this point.

Step 5. The optimal strategy is to produce 3809 of the 19" LCD TV sets and 6117 of the 21" LCD TV sets per year. This will result in about $282,000 profit per year. In the no-tarriff case we made $554,000 per year, so the tarriff has cost the company $272,000 in profits, almost half. Of this amount $248,000 is paid to the government as a tarriff and the remaining $24,000 is due to lost sales.

(b) If we move to the US, net operating costs will increase by $350,000 per year. The optimal production policy in this case is the same as the no-tarriff case, i.e. we should make 4735 of the 19 inch and 7043 of the 21 inch sets. We expect profits of 553,641 - 350,000 = 203,641 dollars per year, and so it would NOT be better to move to the US.

(c) We will use sensitivity analysis to extimate the minimum increase in tarriffs that will make it optimal for the company to relocate to the US. Let r denote the tarriff amount, where we currently assume r = 25 ($ / set).

```
>   unassign('x','y');
> z := (339- .01*x - .003*y)*x +(399 - .004*x - .01*y)*y-(400000 +
  195*x + 225*y + r*(x + y));
```

$$z := (339 - .01\,x - .003\,y)\,x + (399 - .004\,x - .01\,y)\,y - 400000 - 195\,x - 225\,y - r\,(x + y)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := -.02\,x + 144. - .007\,y - r$$

$$dzdy := -.007\,x - .02\,y + 174. - r$$

```
> solve({dzdx=0,dzdy=0},{x,y});
> assign(%);
```

$$\{y = 7042.735043 - 37.03703704\,r,\ x = 4735.042735 - 37.03703704\,r\}$$

```
> dzr:=diff(z,r);
```

$$dzr := -11777.77778 + 74.07407414\,r$$

```
> assign('r=25');
> szr:=dzr*(r/z);
```

$$szr := -.8788835876$$

Then S(z, r) = -0.88 and so if the tariff goes up by 10% the company's expected profits decrease by 8.8%. To motivate the company to relocate we have to decrease the expected profit from 282,000 to 204,000 which is a decline of 28%. This requires a tariff increase of 32% to $33 per set. Since this is just an estimate, and since we want the US relocation to be a better option, not just an equally good option, we recommend setting the tariff at $35 per set or more.

(d) We have already calculated S(z, r) = -0.88 and the following calculations show that S(x, r) = -0.24 while S(y, r) = -0.15. Given that the only motivation of the tariff is to induce the company to relocate to the US, we should set the tariff as low as possible, given that the tariff is large enough to motivate the move. So we recommend a tariff of $35 per set. The only significant result of a tariff higher than $35 is to lower company profits.

```
> unassign('r');
> dxr:=diff(x,r);dyr:=diff(y,r);
```

$$dxr := -37.03703704$$

$$dyr := -37.03703704$$

```
> assign('r=25');sxr:=dxr*(r/x);syr:=dyr*(r/y);
```

$$sxr := -.2430815258$$

$$syr := -.1513740103$$

6. (a) Step 1: Ask the question.

Variables: p = price ($ / computer)
s = sales (computers / month)
a = advertising budget ($ / month)
C = manufacturing cost ($ / month)
R = revenue ($ / month)
P = profit ($ / month)

Assumptions: C = 700 s + a
s = 10,000 + 5000 (950 - p) / 100 + 200 (a - 50000) / 10000
R = p s
P = R - C
p >= 0 and 0 <= a <= 100,000

Objective: Maximize P.

Step 2: Select the modeling approach.

We will model this problem as a multivariable constrained optimization problem, and we will solve the problem using the method of Lagrange multipliers. See text p. 33.

Step 3: Formulate the model.

Let x = p, y = a, and z = P, and write
z = f(x, y) = (x - 700) (10,000 + 5000 (950 - x) / 100 + 200 (y - 50000) / 10000) - y
Our goal is to maximize f(x, y) over the set of all (x, y) for which x >= 0 and 0 <= y <= 10,000.

Step 4: Solve the model.

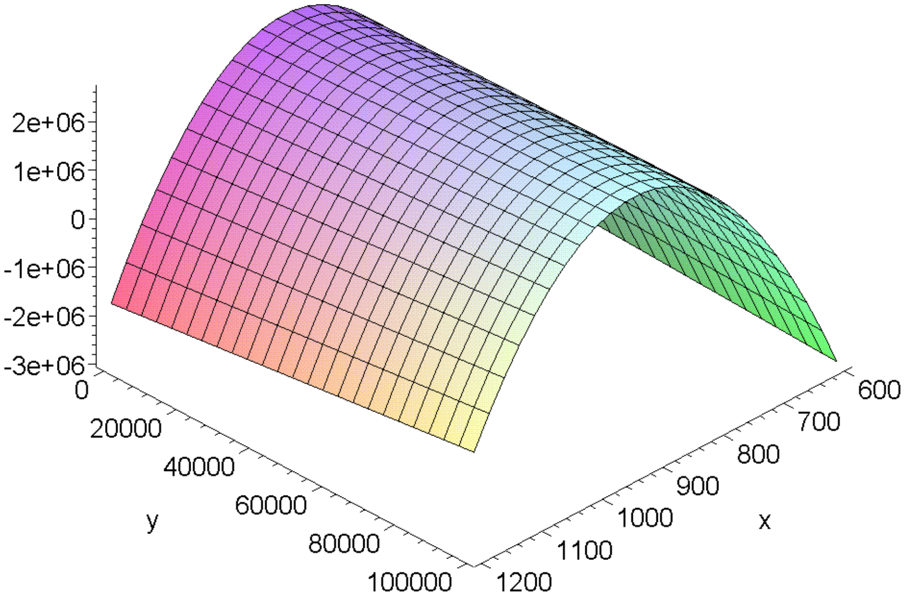
First we look for possible maxima in the interior of the feasible region.

```
> unassign('x','y');
```

```
> z:= (x - 700)*(10000 + 5000*(950 - x) / 100 + 200*(y - 50000) /
  10000) - y;
```

$$z := (x - 700)\left(56500 - 50\,x + \frac{1}{50}\,y\right) - y$$

```
> plot3d(z, x=600..1200,y=0..100000,axes=framed);
```



```
> dzdx:=diff(z,x);
```

$$dzdx := 91500 - 100\,x + \frac{1}{50}\,y$$

```
> dzdy:=diff(z,y);
```

$$dzdy := \frac{1}{50}\,x - 15$$

```
> solve({dzdx=0,dzdy=0},{x,y});
```

$$\{x = 750,\ y = -825000\}$$

Since the only point at which the gradient is zero is outside the feasible region, there are no local extrema in the interior. Now look along the constraint line g(x, y) = y = 100,000. Note that the gradient of g is (0,1).

```
> solve({dzdx=0,dzdy=lambda,y=100000},{x,y,lambda});
```

$$\{y = 100000,\ x = 935,\ \lambda = \frac{37}{10}\}$$

```
> assign(%);
> z;
```

$$2661250$$

The maximum occurs when x = 935. At this point y = 100,000 and z = 2,661,250. Also the value of the Lagrange multiplier is 3.7. Note that setting y = 100,000 yields a quadratic function of x alone, and this is the vertex of that parabola, so this is the maximum on the line y = 100,000. The graph of f(x, y) is very flat in the y direction, but it is easy to see that z is negative for large x. On the line g(x, y) = y = 0 we similarly find

```
> unassign('x','y','lambda');
> solve({dzdx=0,dzdy=lambda,y=0},{x,y,lambda});
> assign(%);
```

$$\{y=0,\ \lambda=\frac{33}{10},\ x=915\}$$

```
> z;
```

$$2311250$$

that the maximum occurs at x = 915, y = 0, and z = 2,311,250 which is less than the maximum along y = 100,000. On the line g(x, y) = x = 0 we have gradient vector (1,0) and

```
> unassign('x','y','lambda');
> assign(x=0);
> dzdx;
```

$$91500+\frac{1}{50}y$$

```
> dzdy;
```

$$-15$$

there are no relative extrema on this segment of the boundary. Since dz/dy < 0 the maximum on this segment occurs at the origin. At this point z < 0. So the maximum over the entire feasible region is at the point x = 935, y = 100,000 and z = 2,661,250.

Step 5. We recommend lowering the price slightly to $935 and spending $100,000 per month on advertising. This should result in a monthly profit of around $2,660,000.

(b) Let e denote the price elasticity, where currently we assume e = (5000 / 100) = 50. In other words, every $1 drop in price is assumed to generate 50 more sales per month. Now

z = f(x, y) = (x - 700) (10,000 + e (950 - x) + 200 (y - 50000) / 10000) - y

and we need to maximize f(x, y) over the set of all (x, y) for which x >= 0 and 0 <= y <= 10,000. For values of e near 50 the maximum should still occur on the constraint line y = 100,000 and so

```
> unassign('x','y','lambda');
> z:= (x - 700)*(10000 + e*(950 - x) + 200*(y - 50000) / 10000) - y;
```

$$z := (x-700)\left(9000+e\,(950-x)+\frac{1}{50}y\right)-y$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := 9000+e\,(950-x)+\frac{1}{50}y-(x-700)\,e$$

$$dzdy := \frac{1}{50}x-15$$

```
> solve({dzdx=0,dzdy=lambda,y=100000},{x,y,lambda});
```

$$\{y = 100000,\ \lambda = \frac{1}{2}\,\frac{220 + 3\,e}{e},\ x = 275\,\frac{3\,e + 20}{e}\}$$

```
> assign(%);
> dxde:=diff(x,e);
```

$$dxde := 825\,\frac{1}{e} - \frac{275\,(3\,e + 20)}{e^2}$$

```
> assign('e=50');
> sxe:=dxde*(e/x);
> evalf(%\);
>
```

$$sxe := \frac{-2}{17}$$

$$-.1176470588$$

we find that S(x, e) = -0.117 and since we are on the constraint line y = 100,000 we have S(y, e) = 0. Both the optimal selling price and the optimal advertising expenditure are quite insensitive to the price elasticity e. If the effect of lowering price is 10% greater than expected (i.e. if lowering the price by $100 results in a 55% increase in sales) then the optimal selling price is about 1.1% lower than expected, and the optimal advertising budget stays the same.

(c) Let n denote the number of new sales for each $10,000 increase in the ad budget. Currently we assume n = 200. Now

z = f(x, y) = (x - 700) (10,000 + 50 (950 - x) + n (y - 50000) / 10000) - y

and we need to maximize f(x, y) over the set of all (x, y) for which x >= 0 and 0 <= y <= 10,000. For values of n near 200 the maximum should still occur on the constraint line y = 100,000 and so

```
> unassign('x','y','lambda');
> z:= (x - 700)*(10000 +50*(950 - x) +n*(y - 50000) / 10000) - y;
```

$$z := (x - 700)\left(57500 - 50\,x + \frac{1}{10000}\,n\,(y - 50000)\right) - y$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := 92500 - 100\,x + \frac{1}{10000}\,n\,(y - 50000)$$

$$dzdy := \frac{1}{10000}\,(x - 700)\,n - 1$$

```
> solve({dzdx=0,dzdy=lambda,y=100000},{x,y,lambda});
```

$$\{y = 100000,\ \lambda = -1 + \frac{9}{400}\,n + \frac{1}{200000}\,n^2,\ x = 925 + \frac{1}{20}\,n\}$$

```
> assign(%);
> dxdn:=diff(x,n);
> assign('n=200');
```

```
> sxn:=dxdn*(n/x);
> evalf(%);
```

$$dxdn := \frac{1}{20}$$

$$sxn := \frac{2}{187}$$

$$.01069518717$$

we find that S(x, e) = -0.01 and since we are on the constraint line y = 100,000 we have S(y, e) = 0. Both the optimal selling price and the optimal advertising expenditure are quite insensitive to n. Overall, the decision to leave the price at around $950 and to increase the advertising budget to $100,000 seems nearly optimal regardless of the true value of n and e, assuming they are at least close to what we have assumed.

(d) The value of the Lagrange multiplier was 3.7 which means that every $1 increase in the advertising budget should result in about $3.70 in additional profits. Management should probably consider a larger advertising budget, assuming that the effect of advertising is as great as the ad agency claims.

7. (a) Step 1: Ask the question.

Variables: p = subscription price ($ / paper)
a = advertising price ($ / page)
s = advertising sales (pages)
c = circulation (papers)
R = weekly revenue ($)

Assumptions: $s = 350 + 50\,(250 - a) / 100$
$c = 80{,}000 + 5000\,(1.50 - p) / 0.10 + 1000\,(s - 350) / 50$
$R = p\,c + a\,s$
$p >= 0$, $a >= 0$

Objective: Maximize R.

Step 2: Select the modeling approach.

We will model this problem as a multivariable unconstrained optimization problem. See text p. 23.

Step 3: Formulate the model.

Let x = p, y = a, and z = R, and write

$$z = f(x, y) = x\,(80{,}000 + 5000\,(1.50 - x) / 0.10 + 1000\,(50\,(250 - y) / 100) / 50)$$

+ y (350 + 50 (250 - y) / 100)

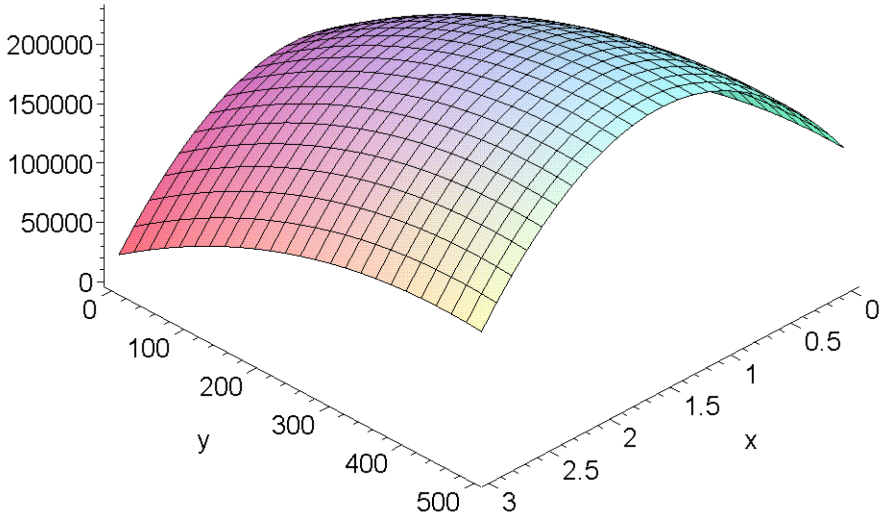
Our goal is to maximize f(x, y) over the set of all (x, y) for which x >= 0 and y >= 0.

Step 4: Solve the model.

```
> unassign('x','y');
> z:=x*(80000+5000*(1.50-x)/0.10+1000*(50*(250-y)/100)/50)+y*(350+50
  *(250-y)/100);
```

$$z := x\,(157500.0000 - 50000.00000\,x - 10\,y) + y\left(475 - \frac{1}{2}\,y\right)$$

```
> plot3d(z, x=0..3,y=0..500,axes=framed);
```



```
> dzdx:=diff(z,x);
```

$$dzdx := 157500.0000 - 100000.0000\,x - 10\,y$$

```
> dzdy:=diff(z,y);
```

$$dzdy := -10\,x + 475 - y$$

```
> solve({dzdx=0,dzdy=0},{x,y});
```

$$\{x = 1.529029029,\ y = 459.7097097\}$$

```
> assign(%);
> z;
```

$$229592.0921$$

The graph indicates a unique maximum at the point where the gradient is zero. We calculate that this occurs when x = 1.53, y = 459.71 and z = 229,592.10.

Step 5. Our model suggests that we leave the subscription price at $1.50 per week and raise the advertising price to around $450 per page. This should result in weeky revenue (profit) of around $230,000.

(b) We generalize the model from part (a) and assume that a 10 cent increase in subscription price results in n lost sales, where currently n = 5000.

```
> unassign('x','y','n');
> z:=x*(80000+n*(1.50-x)/0.10+1000*(50*(250-y)/100)/50)+y*(350+50*(250-y)/100);
```

$$z := x\,(82500 + 10.00000000\,n\,(1.50 - x) - 10\,y) + y\left(475 - \frac{1}{2}\,y\right)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := 82500 + 10.00000000\,n\,(1.50 - x) - 10\,y - 10.00000000\,n\,x$$

$$dzdy := -10\,x + 475 - y$$

```
> solve({dzdx=0,dzdy=0},{x,y});
> assign(%);
```

$$\{x = .2500000000\,\frac{15550. + 3.\,n}{n - 5.},\; y = 27.50000000\,\frac{-1500. + 17.\,n}{n - 5.}\}$$

```
> z;
```

$$.2500000000\,(15550. + 3.\,n)\left(82500 + 10.00000000\,n\left(1.50 - \frac{.2500000000\,(15550. + 3.\,n)}{n - 5.}\right) - \frac{275.0000000\,(-1500. + 17.\,n)}{n - 5.}\right) / (n - 5.) + \frac{27.50000000\,(-1500. + 17.\,n)\left(475 - \frac{13.75000000\,(-1500. + 17.\,n)}{n - 5.}\right)}{n - 5.}$$

```
> dxdn:=diff(x,n):
> dydn:=diff(y,n):
> dzdn:=diff(z,n):
> assign(n=5000);
> sxn:=dxdn*(n/x);
```

$$sxn := -.5100026378$$

```
> syn:=dydn*(n/y);
```

$$syn := .01696307090$$

```
> szn:=dzdn*(n/z);
```

$$szn := -.009666323346$$

Then S(x, n) = -0.51, S(y, n) = +0.017, and S(z, n) = -0.01 so that only the optimal subscription price is sensitive to this parameter. For example, if a 10 cent increase in subscription price actually results in a loss of 6000 subscriptions (20% more than expected) then the optimal subscription price should be about 11% lower than the calculated optimum of 1.53, or around $1.35.

(c) We generalize the model from part (a) and assume that a $100 per page increase in the price of advertising results in m pages per week of lost sales, where currently m = 50.

```
> unassign('x','y','m');
> z:=x*(80000+5000*(1.50-x)/0.10+1000*(m*(250-y)/100)/50)+y*(350+m*(
  250-y)/100);
```

$$z := x\left(155000.0000 - 50000.00000\,x + \frac{1}{5}\,m\,(250-y)\right) + y\left(350 + \frac{1}{100}\,m\,(250-y)\right)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := 155000.0000 - 100000.0000\,x + \frac{1}{5}\,m\,(250-y)$$

$$dzdy := -\frac{1}{5}\,x\,m + 350 + \frac{1}{100}\,m\,(250-y) - \frac{1}{100}\,y\,m$$

```
> solve({dzdx=0,dzdy=0},{x,y});
```

$$\{y = 250.\,\frac{-21900.\,m - .3500000\;10^{7} + m^{2}}{m\,(m - 50000.)},\ x = -12.50000000\,\frac{6060. + m}{m - 50000.}\}$$

```
> assign(%);
> z;
```

$$-12.50000000\,(6060. + m)\Bigg(155000.0000 + \frac{625000.0000\,(6060. + m)}{m - 50000.} + \frac{1}{5}\,m\left(250 - \frac{250.\,(-21900.\,m - .3500000\;10^{7} + m^{2})}{m\,(m - 50000.)}\right)\Bigg)\Big/(m - 50000.) + 250.\,(-21900.\,m - .3500000\;10^{7} + m^{2})\left(350 + \frac{1}{100}\,m\left(250 - \frac{250.\,(-21900.\,m - .3500000\;10^{7} + m^{2})}{m\,(m - 50000.)}\right)\right)\Big/(m\,(m - 50000.))$$

```
> dxdm:=diff(x,m):
> dydm:=diff(y,m):
> dzdm:=diff(z,m):
> assign(m=50);
> sxm:=dxdm*(m/x);
> sym:=dydm*(m/y);
> szm:=dzdm*(m/z);
```

$$sxm := .009184307056$$

$$sym := -.7616555042$$

$$szm := -.2239158881$$

Then S(x, m) = +0.01, S(y, m) = -0.76, and S(z, m) = -0.22 so that both the optimal ad price and the optimal revenue (profit) are sensitive to this parameter, but the optimal subscription price is not.
If the negative reaction to an increase in ad prices is 10% greater than expected, then ad sales will be 7.6% lower than expected and total revenue will be 2.2% lower than expected.

(d) In order to prevent advertisers from switching to direct mail, we might want to charge less than the optimum of $450 per page, which is close to the $500 per page direct mail charge. See problem 8 below for the case where we only raise the ad price to $400 per page.

8. (a) Step 1 is the same as problem 7 above, except that now we assume that the advertising price a <= 400 ($ / page). Step 2 is to solve the model using Lagrange multipliers. Step 3 is to let x = p, y = a, and z = R, and write

z = f(x, y) = x (80,000 + 5000 (1.50 - x) / 0.10
+ 1000 (50 (250 - y) / 100) / 50)
+ y (350 + 50 (250 - y) / 100)

Our goal is to maximize f(x, y) over the set of all (x, y) for which x >= 0 and 0 <= y <= 400. As for step 4, we know from problem 7 that there are no local extrema in the interior of the feasible region. Also see problem 7 for a graph of the objective function. Consider the boundary line g(x, y) = y = 400. Then the gradient of g is (0,1) and we calculate

```
> unassign('x','y','lambda');
> z:=x*(80000+5000*(1.50-x)/0.10+1000*(50*(250-y)/100)/50)+y*(350+50
  *(250-y)/100);
```

$$z := x\,(157500.0000 - 50000.00000\,x - 10\,y) + y\left(475 - \frac{1}{2}\,y\right)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := 157500.0000 - 100000.0000\,x - 10\,y$$

$$dzdy := -10\,x + 475 - y$$

```
> solve({dzdx=0,dzdy=lambda,y=400},{x,y,lambda});
> assign(%);
```

$$\{y = 400.,\ x = 1.535000000,\ \lambda = 59.65000000\}$$

```
> z;
```

$$227811.2500$$

so that the maximum on this segment of the boundary occurs at x = 1.54, y = 400, and z = 227,811. The value of the Lagrange multiplier is 59.65. On g(x, y) = y = 0 we find

```
> unassign('x','y','lambda');
> solve({dzdx=0,dzdy=lambda,y=0},{x,y,lambda});
```

$$\{y = 0.,\ x = 1.575000000,\ \lambda = 459.2500000\}$$

```
> assign(%);
```

```
> z;
```

$$124031.2500$$

a maximum at x = 1.575, y = 0, z = 124031 and on the line g(x, y) = x = 0 we find

```
> unassign('x','y','lambda');
> solve({dzdx=lambda,dzdy=0,x=0},{x,y,lambda});
> assign(%);
```

$$\{ y = 475., \lambda = 152750., x = 0. \}$$

```
> z;
```

$$112812.5000$$

a maximum at x = 0, y = 475, z = 112813. Since z < 0 when x is sufficiently large, the global maximum occurs at x = 1.54, y = 400, and z = 227,811. Step 5: If we wish to keep advertising prices at or below $400 per page, we should set the ad price at $400 per page and increase subscription price by about 5 cents per week. This will result in total revenues of about $227,800 per week. This is only $2,200 less than the projected maximum of $229,600 found in problem 7, and it will protect the newspaper from potential loss of advertising customers.

(b) We generalize the model from part (a) and assume that a 10 cent increase in subscription price results in n lost sales, where currently n = 5000.

```
> unassign('x','y','n','lambda');
> z:=x*(80000+n*(1.50-x)/0.10+1000*(50*(250-y)/100)/50)+y*(350+50*(2
  50-y)/100);
```

$$z := x\,(82500 + 10.00000000\,n\,(1.50 - x) - 10\,y) + y\left(475 - \frac{1}{2}\,y\right)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := 82500 + 10.00000000\,n\,(1.50 - x) - 10\,y - 10.00000000\,n\,x$$

$$dzdy := -10\,x + 475 - y$$

```
> solve({dzdx=0,dzdy=lambda,y=400},{x,y,lambda});
```

$$\{ y = 400., x = .2500000000\,\frac{15700. + 3.\,n}{n}, \lambda = 2.500000000\,\frac{-15700. + 27.\,n}{n} \}$$

```
> assign(%);
> dxdn:=diff(x,n);
> dydn:=diff(y,n);
```

$$dxdn := -.2500000000\,\frac{15700. + 3.\,n}{n^2} + \frac{.7500000000}{n}$$

$$dydn := 0$$

```
> assign(n=5000);
> sxn:=dxdn*(n/x);
> syn:=dydn*(n/y);
```

$$sxn := -.5114006515$$

$$syn := 0.$$

Then S(x, n) = -0.51, and S(y, n) = 0 so that only the optimal subscription price is sensitive to this parameter. For example, if a 10 cent increase in subscription price actually results in a loss of 6000 subscriptions (20% more than expected) then the optimal subscription price should be about 11% lower than the calculated optimum of 1.54, or around $1.35.

(c) We generalize the model from part (a) and assume that a $100 per page increase in the price of advertising results in m pages per week of lost sales, where currently m = 50.

```
> unassign('x','y','m','lambda');
> z:=x*(80000+5000*(1.50-x)/0.10+1000*(m*(250-y)/100)/50)+y*(350+m*(
  250-y)/100);
```

$$z := x\left(155000.0000 - 50000.00000\,x + \frac{1}{5}\,m\,(250-y)\right) + y\left(350 + \frac{1}{100}\,m\,(250-y)\right)$$

```
> dzdx:=diff(z,x);
> dzdy:=diff(z,y);
```

$$dzdx := 155000.0000 - 100000.0000\,x + \frac{1}{5}\,m\,(250-y)$$

$$dzdy := -\frac{1}{5}\,x\,m + 350 + \frac{1}{100}\,m\,(250-y) - \frac{1}{100}\,y\,m$$

```
> solve({dzdx=0,dzdy=lambda,y=400},{x,y,lambda});
> assign(%);
>
```

$$\{y = 400.,\ \lambda = 350. - 5.810000000\,m + .00006000000000\,m^2,$$
$$x = 1.550000000 - .0003000000000\,m\}$$

```
> dxdm:=diff(x,m);
```

$$dxdm := -.0003000000000$$

```
> assign(m=50);
> sxm:=dxdm*(m/x);
```

$$sxm := -.009771986971$$

Then S(x, m) = -0.01, and of course S(y, m) = 0 so that neither decision variable is sensitive to the exact magnitude of negative reaction to an increase in ad prices. This makes intuitive sense since we have deliberately cushioned against this negative impact.

(d) The value of the Lagrange multiplier is 59.65. This is also the derivative of total weekly revenue z ($) with respect to the upper bound c on ad prices, where currently we assume c = 400 ($ / page). For example, if we raise this upper bound by $10 per page then revenue would increase by $596.50 per week. This loss protects the newspaper against a potential loss in advertising to the direct mail competition. Roughly speaking, every $1 drop in advertising price costs the company $60 in profits.

9. (a) Step 1: Ask the question.

Variables:
s = advertising sales (pages)
c = circulation (papers)
E = weekly editorial expense ($)
B = weekly sales budget ($)
R = weekly revenue ($)
C = weekly cost ($)
P = weekly profit ($)

Assumptions:
$s = 350 + (.01*350)(E - 80{,}000) / 8{,}000 + (0.15*350)(B - 30{,}000) / 6000$
$c = 80{,}000 + (.02*80{,}000)(E - 80{,}000) / 8{,}000$
$R = 1.50\,c + 250\,s$
$C = E + B + 90{,}000$
$P = R - C$
$40{,}000 \le E \le 80{,}000$
$30{,}000 \le B \le 50{,}000$
$C \le 200{,}000$

Objective: Maximize P.

Step 2: Select the modeling approach.

We will model this problem as a multivariable constrained optimization problem, and we will solve the problem using the method of Lagrange multipliers. See text p. 33.

Step 3: Formulate the model.

Let $x = E$, $y = B$, and $z = P$, and write

$$\begin{aligned} z = f(x, y) = {} & 1.50\,(80{,}000 + (.02*80{,}000)(x - 80{,}000) / 8{,}000) \\ & + 250\,(350 + (.01*350)(x - 80{,}000) / 8{,}000 + (0.15*350)(y - 30{,}000) / 6000) \\ & - (x + y + 90{,}000) \end{aligned}$$

Our goal is to maximize $f(x, y)$ over the set S of all (x, y) for which
$40{,}000 \le x \le 80{,}000$
$30{,}000 \le y \le 50{,}000$
$x + y \le 110{,}000$

Step 4: Solve the model.

The objective function is linear and so the gradient of $f(x, y)$ is never zero. Then there are no interior extreme points. The restriction of $f(x, y)$ to a line is still linear, so there are no local extrema along any of the line segments bounding the feasible region. Then the maximum must occur at one of the corners. We check each corner to determine the maximum

```
> unassign('x','y');
> z:=1.50*(80000+(.02*80000)*(x-80000)/8000)
  +250*(350+(.01*350)*(x-80000)/8000+(0.15*350)*(y-30000)/6000)-(x+y
  +90000);
                    z := 19125.0000 – .5906250000 x + 1.187500000 y
> assign({x=40000,y=30000});z;
                              31125.00000
> unassign('x','y');assign({x=40000,y=50000});z;
                              54875.00000
> unassign('x','y');assign({x=80000,y=30000});z;
                               7500.00000
> unassign('x','y');assign({x=60000,y=50000});z;
                              43062.50000
```

The maximum occurs at (x, y) = (40000, 50000) where z = 54875.

Step 5. We recommend cutting editorial expense to $40,000 and increasing the ad budget to $50,000 per week. This should increase profits dramatically from their current level of $7,500 per week to around $55,000 per week.

(b) The binding constraints are g1(x,y) = x = 40,000 and g2(x,y) = y = 50,000. The gradient vectors are (1,0) and (0,1) respectively, and so the Lagrange multiplier equation is
(dz/dx,dz/dy) = lambda1 (1,0) + lambda2 (0,1). At the optimum we have
lambda1 = -0.59 and lambda2 = 1.1875. All other things being equal, a change in the maximum ad budget from 50,000 to 50,001 would net the company $1.19. All other things being equal, a change in the minimum editorial budget from 40,000 to 40,001 would cost the company 59 cents. The effects are additive, so that if the ad budget is increased by one dollar and the editorial budget is decreased by one dollar, then the change in net profit is 1.19 + 0.59 = 1.78 dollars. The constraints which give lower bounds on the ad budget, upper bounds on the editorial budget, and upper bounds on the total budget are not binding, and so their associated shadow prices are zero. Small changes in these figures will not effect net profit.

(c) The feasible region is the rectangle defined by

40,000 <= x <= 80,000
30,000 <= y <= 50,000

intersected with the half-plane x + y <= 110,000. The optimum is on the upper left hand corner where the lines y = 50,000 and x = 40,000 intersect. The shadow prices are nonzero for exactly these two binding constraints.

(d) Generalize the model from part (a) so that now

```
> unassign('x','y');
> z:=1.50*(80000+(2*p*80000)*(x-80000)/8000)
  +250*(350+(p*350)*(x-80000)/8000+(0.15*350)*(y-30000)/6000)-(x+y+9
```

```
0000);
```

$$z := 51875.0000 + 40.93750000\, p\, (x - 80000) + 1.187500000\, y - x$$

```
> assign({x=40000,y=30000});z1:=z;
```

$$z1 := 47500.00000 - .1637500000\; 10^{7}\, p$$

```
> unassign('x','y');assign({x=40000,y=50000});z2:=z;
```

$$z2 := 71250.0000 - .1637500000\; 10^{7}\, p$$

```
> unassign('x','y');assign({x=80000,y=30000});z3:=z;
```

$$z3 := 7500.00000$$

```
> unassign('x','y');assign({x=60000,y=50000});z4:=z;
```

$$z4 := 51250.0000 - 818750.0000\, p$$

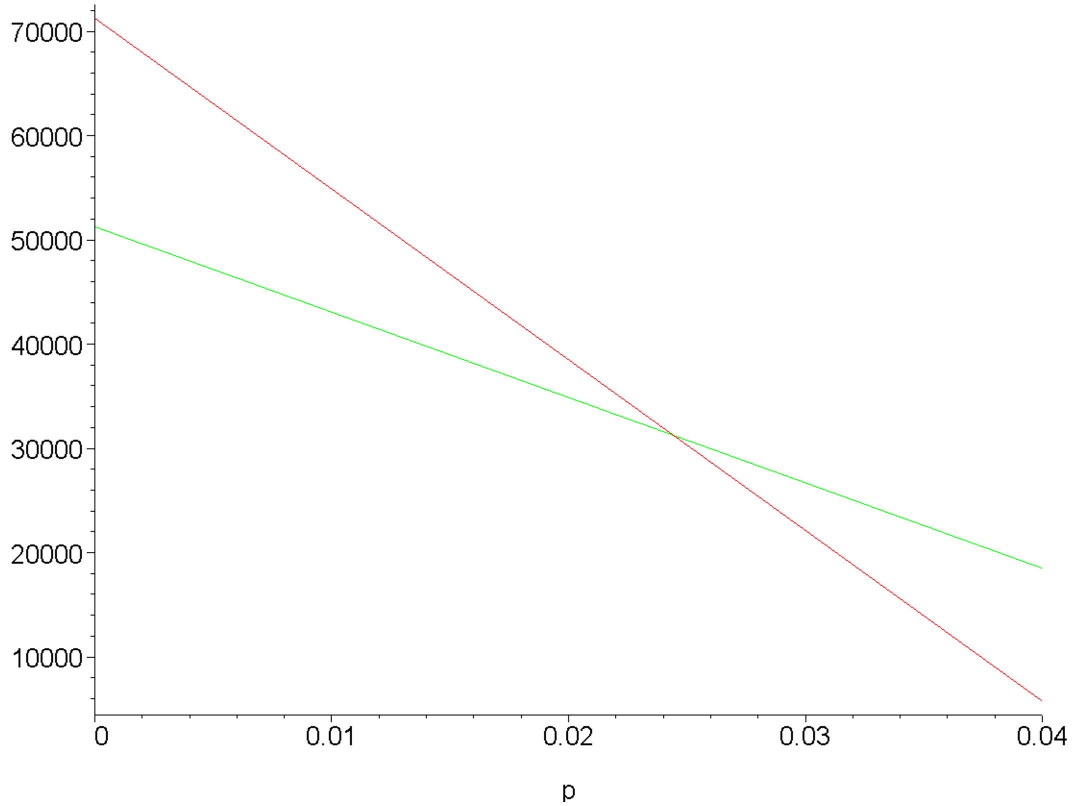
```
> solve({z2=z4},{p});
```

$$\{ p = .02442748092 \}$$

```
> solve({z3=z4},{p});
```

$$\{ p = .05343511450 \}$$

```
> plot({z2,z4},p=0..0.04);
```



and note that z2 > z1 for any p. Then if $p < 0.024$ or 2.4% it is optimal to cut the editorial budget all the way to the minimum of $40,000 per week. If $p > 0.024$ then it is optimal to cut the editorial budget to $60,000 per week. It is never optimal to maintain the editorial budget at its current level of $80,000 per week. Geometrically, the optimum occurs where the level set $z = c$ crosses the feasible region for the largest value of c. For $p < 0.02$ this level set is a line with positive slope, and for $p > 0.024$ this is a line with a negative slope. Increasing $c > 0$ moves the line upward. Thus the only possible maxima for any $p > 0$ are the upper corner points of the feasible region.

10. (a) Step 1: Ask the question.

Variables: x1 = cargo 1 (tons)
x2 = cargo 2 (tons)
x3 = cargo 3 (tons)
W = total weight (tons)
V = total volume (cubic feet)
T = total freight charges ($)

Assumptions: W = x1 + x2 + x3
V = 550 x1 + 800 x2 + 400 x3
T = 250 W
W <= 100
V <= 50,000
0 <= x1 <= 30
0 <= x2 <= 40
0 <= x3 <= 50

Objective: Maximize T.

Step 2: Select the modeling approach.

We will model this problem as a multivariable constrained optimization problem, and we will solve the problem using the method of Lagrange multipliers. See text p. 33.

Step 3: Formulate the model.

Let z = T, and write

z = f(x1, x2, x3) = 250 x1 + 250 x2 + 250 x3

Our goal is to maximize f(x, y) over the set S of all (x, y) for which

x1 + x2 + x3 <= 100
550 x1 + 800 x2 + 400 x3 <= 50,000
0 <= x1 <= 30
0 <= x2 <= 40
0 <= x3 <= 50

Step 4: Solve the model.

The objective function is linear and so the gradient of f(x, y) is never zero. Then there are no interior extreme points. The restriction of f(x, y) to a plane or a line is still linear, so there are no local extrema along any of the faces or the edgesof the feasible region. Then the maximum must occur at one of the corners. We check each corner to determine the maximum. It is not hard to see that it cannot be optimal to set any of x1, x2, x3 equal to zero. Then there are five remaining linear constraints

g1(x1, x2, x3) = x1 + x2 + x3 = 100
g2(x1, x2, x3) = 550 x1 + 800 x2 + 400 x3 = 50,000
g3(x1, x2, x3) = x1 = 30
g4(x1, x2, x3) = x2 = 40
g5(x1, x2, x3) = x3 = 50

and by solving any 3 of these we get the coordinates of a corner point. If these coordinates satisfy the inequality constraints then the corner point represents a feasible solution. By checking all 10corner points we find that the optimum occurs at the intersection of constraint lines g2, g3, and g5 given by (x1, x2, x3) = (30, 16.875, 50) at which we find z = 24218.75. For example

```
> unassign('x1','x2','x3');
```

```
> solve({550*x1+800*x2+400*x3=50000,x1=30,x3=50},{x1,x2,x3});evalf(%);
```

$$\{x1 = 30,\ x3 = 50,\ x2 = \frac{135}{8}\}$$

$$\{x1 = 30.,\ x3 = 50.,\ x2 = 16.87500000\}$$

```
> assign(%);z:=250*x1+250*x2+250*x3;
```

$$z := 24218.75000$$

Step 5. The optimal strategy is to ship the maximum of 30 tons/day of cargo 1 and 50 tons/day of cargo 3. Then because of volume constraints we can only ship 16.875 tons/day of cargo 2. This will yield a total shipping charge of $24,218.75 per day. The weight constraint is not binding, i.e. we do not have enough volume in the cargo holds to ship all 100 tons of available cargo.

(b) The gradient vectors for the binding constraints are v2 = (550, 800, 400) for g2, v3 = (1,0,0) for g3, v5 = (0,0,1) for g5, and w = (250, 250, 250) for the objective function f. The Lagrange multiplier equations are w = lambda2 v2 + lambda3 v3 + lambda5 v5.

```
> solve({250=lambda2*550+lambda3,250=lambda2*800,250=lambda2*400+lambda5},{lambda2,lambda3,lambda5});
```

$$\{\lambda 2 = \frac{5}{16},\ \lambda 5 = 125,\ \lambda 3 = \frac{625}{8}\}$$

```
> evalf(%);
```

$$\{\lambda 2 = .3125000000,\ \lambda 5 = 125.,\ \lambda 3 = 78.12500000\}$$

so that lambda2 = 0.3125, lambda3 = 78.125, and lambda5 = 125. Additional cargo capacity is worth $0.31 per cubic foot. The net advantadge of being able to ship more of cargo 1 is $78.13 per ton, and for cargo 3 the figure is $125 per ton. For all of the other constraints, which are all nonbinding, the shadow prices are zero. So for example the company would not be willing to pay to increase the weight capacity of the planes, since the current optimal solution does not use all of the available weight capacity.

(c) Using the sensitivity results from part (b) above, we know that additional cargo space is worth

$0.31 per cubic foot. Over the useful lifetime of the planes, the proposed modification would allow the company to ship 5*250*2000=2,500,000 additional cubic feet of cargo, which should be worth 0.31*2,500,000 = 775,000 dollars. This is well above the $200,000 cost of the reconfiguration, so the company should proceed with the plan.

The remaining cargo not currently being shipped consists of 40 - 16.875 = 23.125 tons per day of cargo 2, which would fill a volume of 23.125*800 = 18500 cubic feet. Our currrent daily load uses all of the available volume and weighs 96.875 tons. Each modified plane can carry an additional 2000 cubic feet of cargo 2, which weighs 2000/800 = 2.5 tons. Although there is enough additional cargo to fill 9 or 10 modified planes, there is only enough total weight capacity to carry an additional 3.125 tons per day, so we should only modify one or two of the planes. If we modify one plane then the binding constraints are g3, g5, and the modified constraint g2 <= 52,000.

```
> unassign('x1','x2','x3');
> solve({550*x1+800*x2+400*x3=52000,x1=30,x3=50},{x1,x2,x3});evalf(%
  );
>
```

```
              155
      { x2 = -----, x1 = 30, x3 = 50 }
               8

{ x2 = 19.37500000, x3 = 50., x1 = 30. }
```

```
> assign(%);z:=250*x1+250*x2+250*x3;
```

```
z := 24843.75000
```

and so in this case we ship 19.375 tons of cargo 2 and obtain $24,843.75 per day in shipping charges.
If we modify 2 planes then g3, g5, and g1 <= 100 are binding and so we compute

```
> unassign('x1','x2','x3');
> solve({x1+x2+x3=100,x1=30,x3=50},{x1,x2,x3});
```

```
{ x2 = 20, x1 = 30, x3 = 50 }
```

```
> assign(%);z:=250*x1+250*x2+250*x3;
```

```
z := 25000
```

```
>
```

and so in this case we ship 20 tons of cargo 2 and obtain $25,000 per day in shipping charges.
The difference between two planes and one is (25000-24843.75)*5*200 = 156,250 dollars over the five year lifetime of the planes, and this is less than the $200,000 alteration charge, so it is not worth while to modify both planes.