Math 555, Fall 2002 Prof. J. Rauch

Homework 2. Due Tues September 17

1. This is a continuation of problem **1** from the preceding assignment. Find the relation between the radius of the small image circles in u, v space and the radius of the circles in the x, y space which are mapped to them. The relation depends on the central point $\underline{x}, \underline{y}$. **Hint.** You can check your answer against the expected behavior that little circles near the origin should be greatly expanded while little circles far from the origin should shrink. Again you can discover aspects of the result with computer simulation.

31-33/5,13

42-43/4,7

47-48/1,8

54-55/1,3 **Comments for 3.** Note that for part b. the derivative does not exist most of the time but has a nice formula at the rare points where it does exist.

10. My variant of 55/10. In this case, the textbook version is a bit incomrehensible! Suppose that $P(x, y)$ is a polynomial in the variables x and y. Intoduce new variables $z, \overline{z}$ by the relations

$$z = x + iy, \quad \overline{z} = x - iy\,, \qquad x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i}\,.$$

In this way a P defines a polynomial $\tilde{P}$ of the variables $z, \overline{z}$ by

$$\tilde{P}(z, \overline{z}) := P\big((z + \overline{z})/2\,,\, (z - \overline{z})/2i\big)\,, \qquad P(x, y) = \tilde{P}(x + iy\,,\, x - iy)\,.$$

Show that

$$\frac{\partial \tilde{P}}{\partial \overline{z}}\big(z(x, y), \overline{z}(x, y)\big) \;=\; \frac{1}{2}\Big(\frac{\partial P}{\partial x} + i\frac{\partial P}{\partial y}\Big)$$

Hint. Show that it suffices to prove this identity when P is a monomial. Then continue.

Comments. i. This shows that P satisfies the Cauchy-Riemann equations if and only if $\partial\tilde{P}/\partial\overline{z} = 0$. Thus in the expansion

$$\tilde{P} \;=\; \sum_{n,m\geq 0} a_{n,m}\, z^n\, \overline{z}^m$$

no terms with $m > 0$ appear. In this sense, P satisfies the C-R equations if and only if it is a function only of z and not of $\overline{z}$.

ii. The operator

$$\frac{\partial}{\partial \overline{z}} \;:=\; \frac{1}{2}\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big)\,.$$

appears lots of places so it is good to see it at least once in your life. It comes in a pair with

$$\frac{\partial}{\partial z} \;:=\; \frac{1}{2}\Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\Big)\,.$$

Note the confusing way the plus and minuses appear. Derivative with respect to $\overline{z} = x - iy$ gets the **plus** sign! For an analytic function, the $\overline{z}$ derivative vanishes and the z derivative is equal to f'.