Homework 1. Due Tuesday Sept 15

There are two important pieces of background to buttress at the start of a complex variables course. They are ideas from the differential calculus of maps from the plane to itself, and complex numbers. The first chapter is devoted to the latter together with a bit of plane geometry. Much will likely be familiar. The differential calculus is discussed in Tuesday’s class.

The first problems of the first two assignments are from advanced calculus and serve to remind you how partial derivatives can be used to analyse the behavior of functions mapping the plane to itself. They also have a computational and geometric side which is true of much that we will do in this course. The matrix of partial derivatives is sometimes called the Jacobian matrix after Jacobi. Denoting it by $J$ the key relation is that for points $x, y$ near $x, y$ one has

$$(\Delta u, \Delta v) \approx J(x, y) (\Delta x, \Delta y).$$

This fundamental relation is the two dimensional analogue of

$$\Delta y \approx f'(x) \Delta x$$

which is the key relation in differential calculus of functions of one variable.

1. i. For inversion in the unit circle, defined by

$$(x, y) \mapsto (u, v) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right),$$

compute the Jacobian matrix

$$\begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix}.$$

ii. For the mapping

$$(x, y) \mapsto (x - y, x^2 + y^2),$$

Compute the Jacobian matrix.

iii. In the first case verify that the Jacobian is a conformal linear map for all $(x, y)$ and that for most $(x, y)$ the Jacobian of the second is not.

Discussion. This shows that infinitesimal circles are sent to infinitesimal circles. That is a circle of radius $r << 1$ centered at a point $x, y$ is sent to a set which is very close (error $\sim r^2$) to a circle centered at the image of the central point. The images which might have been ellipses with any eccentricity, have eccentricity is equal to 1. If you find this difficult to digest, you can check the result with computer simulations. (Plot the image of small circles, for example using Matlab).

22/2,6,10 (This means problems 2, 6, and 10 on page 22 of the text.)

29/2

33/1,2, 3 (This is really one question.)