1. 243/3.

2. 243/4.

3. 245/19. The answer given is the unique bounded solution. There are many others. Find the bounded solution.

4. 245/18 Variant. Denote by $\Gamma$ the plane $\mathbb{R}^2$ with the segments $-\infty < x \leq -\sigma$ and $\sigma \leq x < \infty$ on the $x$-axis removed as in the figure. Find the unique temperature $u(x, y)$ harmonic and bounded on $\Gamma$ with $u = -1$ on the left hand removed half line and $u = 1$ on the right hand removed half line. **Hint.** We know how to find a bounded harmonic function in the upper half disk that vanishes on the diameter and is equal to one on the top of the disk.

Consider the domain $\Theta$ equal to the unit disk with center at the origin with the interval $-1 < x \leq 0$ on the $x$-axis removed. The map $z^{1/2}$ suitably defined maps $\Theta$ to the right half disk. This allows one to solve the Dirichlet problem on the slit disk $\Theta$ with values 1 on both sides of the slit and value 0 on the circumference of $\Theta$. Send the intersection of the slit with the circumference to infinity to solve a heat flow problem sketched in the third figure. Finish by using symmetry of the original problem with respect to the $y$-axis.

The next three problems concern irrotational, incompressible, inviscid steady planar fluid flows. We call them simply flows.

The next two problems concern flows in the wedge $0 < \theta < A$ where $A < \pi/4$ and the flow is required to be tangent to the bounding half lines. We find the unique complex potentials $F(z)$ defining such flows and having the desirable dilation scaling that for any $\sigma > 0$,

$$F(\sigma z) = c(\sigma) F,$$  \hspace{1cm} (1)
for a suitable real \( c(\sigma) \) depending on \( \sigma \).

5. i. Show that if \( F \) is analytic in the wedge and satisfies (1), then if \( \sigma \) and \( \tau \) are two positive constants then \( c(\sigma \tau) = c(\sigma)c(\tau) \). ii. Show that \( c \) is a differentiable function on \( ]0, \infty[ \). **Hint.** Consider a single fixed \( z \). iii. Show that \( F \) is a homogeneous function of \( z \). **Hint.** Use logarithms to nearly determine \( c(\sigma) \). iv. Show that an analytic \( F \) on the wedge is of the form \( b z^\alpha \) for some real \( \alpha \) and complex \( b \). **Hint.** Read the earlier homework problem identifying analytic functions homogeneous of degree \( n \) with \( n \) integer.

6. Find all flows in the wedge whose flow is parallel to the bounding lines and satisfies the symmetry (1). **Hint.** Use the result of the preceding problem. **Discussion.** The flow velocity is bounded at the corner but the derivatives of the velocity are unbounded at the corner. The case of \( A = \pi/4 \) does not have this divergence.

7. i. Starting with flow in the unit disk swirling about the origin, find a swirling flow in the upper half disk so that the flow swirls about the point midway between the circle center and the circumference. ii. Show that near the corners of the half disk the flow resembles the flow with complex potential \( z^2 \) in a quadrant.