Final Exam, December 19, 2013

Instructions. 1. Closed book. Two 3.5in. × 5in. sheet (four sides) of notes from home.
2. No electronics, phones, cameras, ... etc.
3. Show work and explain clearly.
4. There are 9 questions on 9 pages. 50 points total.

1. (5 points). Suppose that \( R \subset \mathbb{C} \) is a nice bounded domain with boundary oriented in
the standard way. Show that
\[
\oint_{\partial R} x \, dz = i \text{Area}(R).
\]

Hint. This tests the derivation of Cauchy’s Theorem.

Solution. Green’s Theorem that asserts that
\[
\oint_{\partial R} P(x,y) \, dx + Q(x,y) \, dy = \int \int_{R} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx \, dy.
\]
Write \( xdz = x(dx + idy) \) and apply the formula with \( P = x, \ Q = ix \) to find
\[
\oint_{\partial R} x \, dz = \int \int_{R} i \, dx \, dy = i \text{Area}(R).
\]

2. (4 points). Is there a nonempty open disk \( D \) on which the function
\[
\left( \frac{x^2}{2} + xy \right) + i \left( xy + \frac{y^2}{2} \right)
\]
is analytic?

Solution. The Cauchy-Riemann equations assert that \( f_x + if_y = 0 \) for analytic functions.
Compute
\[
\frac{\partial f}{\partial x} = (x+y) + iy, \quad \frac{\partial f}{\partial y} = x + i(x+y).
\]
Therefore
\[
\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0 + i(x+y).
\]
This vanishes only on the line \( x + y = 0 \) and therefore on no nonempty open disk.

Discussion. The Cauchy-Riemann equations at a point don’t yield anything. Analyticity at a point means in a disk with center at the point.
3. (4+2 points). i. Define with integer \( n \geq 1, \)

\[
g(z) := z^n + \frac{1}{z - i}.
\]

For \( R > 2, \) how many times does the image by \( g(z) \) of the circle \( \{ |z| = R \} \) wind around the origin in the positive sense? Explain.

ii. How many roots \( z \) of \( g(z) = 0 \) are there in the complex plane?

**Solution.** i. On \( |z| = R > 1 \) one has

\[
|z^n| = R^n, \quad |1/(z - i)| \leq 1/(R - 1).
\]

For \( R > 2 \) the first is strictly larger than the second.

The dog on a leash principal implies that the image by \( g \) winds that same number of times as the image by \( z^n \). The image by \( z^n \) winds \( n \) times. **Ans.** \( n \).

ii. The argument principal asserts that this winding number is equal to the number of zeroes minus the number of poles.

There is one simple pole at \( z = i \). Therefore

\[
n = \text{number of zeros} - \text{number of poles} = \text{number of zeros} - 1.
\]

There are \( n + 1 \) zeros.

**Alternate.** \( p(z) := (z - i)g(z) = (z - i)z^n + 1 \) is a polynomial of degree \( n + 1 \) so has exactly \( n + 1 \) roots.

Since \( p(i) = 1 \neq 0 \) it follows that \( p \) has exactly \( n + 1 \) roots in \( \mathbb{C} \setminus i \). Therefore \( g \) also has \( n + 1 \) roots in \( \mathbb{C} \setminus i \). Since \( i \) is not a root of \( g \) this counts the roots of \( g \). Note the care needed to treat the danger of division by zero when passing from \( p \) to \( g \).

4. (2+2 points). Define \( \text{arg}(z) \) in the slit plane \( \mathbb{C} \setminus (-\infty, 0] \) by \( -\pi < \text{arg}(z) < \pi \). Define the analytic \( \ln(z) = \ln|z| + i\text{arg}(z) \) on the slit plane. Then \( G(z) := \ln(z) - i\pi \) is analytic in the quadrant \( \{ z = x + iy : x < 0, \ 0 < y \} \). \( G \) is continuous up to the boundary of the quadrant on the \( x \)-axis with real values there.
i. By what formula does $G(z)$ continue analytically to the left half plane $\{x < 0\}$?

ii. Compute the value of the continuation $G(-1 - i)$.

**Solution.**

i. In the left half plane the values of $G(z)$ in $\text{Im} \, z < 0$ are computed from the values in $\text{Im} \, z > 0$ from the formula

$$G(z) = \overline{G(\overline{z})}.$$ 

This identity then holds throughout the left half plane.

ii. Compute

$$G(-1 - i) = 
\overline{G(-1 + i)} = G(-1 + i) = \ln(-1 + i) - i\pi = ((\ln \sqrt{2}) + i3\pi/4) - i\pi = \ln \sqrt{2} - i\pi/4.$$ 

Therefore $G(-1 - i) = \ln \sqrt{2} + i\pi/4$.

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5. (6 points). Evaluate

$$\oint_{|z-3|=1} \frac{\cos z}{\sin z} \, dz.$$ 

**Solution.** The sin function in the denominator has simple zeros at the points $z = n\pi$ for integer $n$. The cos in the numerator does not vanish at these points so they are simple poles of $\cos z/\sin z$.

The contour encloses only the pole at $z = \pi$. The residue theorem implies that

$$\oint_{|z-3|=1} \frac{\cos z}{\sin z} \, dz = 2\pi i \, \text{Res}(\cos z/\sin z, z = \pi).$$ 

Since $\cos \neq 0$ at the simple poles, the residue is given by the value of $\cos(z)/\sin'(z)$ evaluated at $z = \pi$. Since $\sin' = \cos$, the residue is equal to 1.

**Ans.** $2\pi i$.

**Alternate.** The residue can be computed by expanding $\cos$ and $\sin$ in Taylor series about $z = \pi$. Those series are quickly computed using

$$\cos z = \cos((z - \pi) + \pi) = -\cos(z - \pi) = -(1 - \frac{(z - \pi)^2}{2!} + \cdots),$$ 

$$\sin z = \sin((z - \pi) + \pi) = -\sin(z - \pi) = -(z - \pi) - \frac{(z - \pi)^3}{3!} + \cdots).$$

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6. (8 points). Evaluate exactly

$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} \, dx.$$
You may leave the answer as a complicated algebraic expression in complex numbers.

Solution. The function

\[ f(z) = \frac{z^2}{1 + z^4} \]

has poles at the roots of the denominator that are the fourth roots of -1, \((\pm 1 \pm i)/\sqrt{2}\). These poles are on the vertices of a square centered at the origin and sides parallel to the axes.

The integrand is infinitely differentiable on the x axis and for \(|z| > 1\),

\[ |f(z)| \leq \frac{|z|^2}{|z|^4 - 1}. \]

Thus the integral is absolutely convergent by comparison at \(\pm\infty\) with the integral of \(1/x^2\).

Apply the Residue Theorem to the open half disk \(\mathcal{R} := \{ |z| < R, \text{ Im} \ z > 0 \}\) with \(R > 1\).

The function \(f\) is analytic in \(\mathcal{R}\) and on the boundary with the exception of isolated singularities at the poles \((\pm 1 + i)/\sqrt{2}\). The Residue Theorem implies that

\[
\oint_{\partial\mathcal{R}} \frac{z^2}{1 + z^4} \, dz = 2\pi i \left( \text{Res}(f, (1 + i)/\sqrt{2}) + \text{Res}(f, (-1 + i)/\sqrt{2}) \right).
\]

Take the limit \(R \to \infty\). On the circular boundary of length \(\pi R\) one has \(|f| \leq R^2/(R^4 - 1)\) so the integral over that boundary tends to zero. The integral over the part of the boundary on the x-axis converges to the desired integral so

\[
\text{Ans.} = 2\pi i \left( \text{Res}(f, (1 + i)/\sqrt{2}) + \text{Res}(f, (-1 + i)/\sqrt{2}) \right).
\]

Let \(g(z) = z^2\) and \(h(z) = 1 + z^4\). Since \(g\) is non zero and \(h\) has a simple zero at each of the poles the residue at a root \(\omega\) of \(-1\) is given by

\[
\frac{g(\omega)}{h'(\omega)} = \frac{\omega^2}{4\omega^3} = \frac{1}{4\omega} = \frac{\overline{\omega}}{4},
\]

since \(|\omega| = 1\).

Therefore

\[
\text{Ans.} = \frac{2\pi i}{4} \left( \frac{(1 + i)}{\sqrt{2}} + \frac{(-1 + i)}{\sqrt{2}} \right) = \frac{2\pi i}{4} \frac{(-2i)}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}.
\]
7. (5 points). Define the argument to have values in $]-\pi, \pi[$. The sector with angular opening $A$ is the set

$$\left\{ z : 0 < \arg z < A \right\}, \quad 0 < A < \frac{\pi}{2}. $$

A sketch follows.

Find a one to one conformal mapping from the sector to an open disk.

**Solution.** The mapping $z \mapsto z^{\pi/A}$ maps the sector to the upper half plane.

The mapping $z \mapsto 1/(z - i) := k(z)$ maps the upper half plane to the inside of the circle that is the image of the real axis. That is a disk. That answers the question.

Continuing one could identify the circle by noting that it contains $0 = k(\infty)$. And it contains the point $i = k(0)$. The mapping $k(z)$ preserves the imaginary axis. Since the imaginary axis is perpendicular to the $x$ axis their images are also perpendicular. Therefore the circle is perpendicular to the imaginary axis. The circle is $D := \{|z - (i/2)| = 1/2\}$ sketched below.

The mapping

$$F(z) = \frac{1}{z^{\pi/A} - i}$$

maps the sector conformally to the disk $D$.

8. (6 points). Find the unique bounded steady temperature distribution (a.k.a. harmonic function) in the positive quadrant $\{z = x + iy : x > 0 \ y > 0\}$ with temperature equal to $-3$ on the strictly positive $y$-axis, and on the $x$-axis has temperatures equal to $0$ on $]0, 1[\}$
and equal to 7 on the segment $[1, \infty[$.

**Solution.** The map $z \mapsto z^2$ takes the quadrant problem to the corresponding upper halfspace problem with boundary values sketched below.

Define the argument function to take values $0 \leq \arg \leq \pi$ in the upper half plane. Seek a solution of the form

$$A \arg(z - 0) + B \arg(z - 1) + C.$$

On the interval $]1, \infty[$ the two args vanish and one finds that

$$C = 7.$$

On the interval $]0, 1[$ the first arg vanishes and one finds

$$B\pi + C = 0, \quad B = \frac{-7}{\pi}.$$

On the interval $]-\infty, 0[$ one finds

$$A\pi + B\pi + C = -3, \quad A = \frac{-3}{\pi}.$$

The solution of the halfspace problem is

$$\frac{-3}{\pi} \arg(z - 0) + \frac{-7}{\pi} \arg(z - 1) + 7.$$

A (and therefore the) solution of the quadrant problem is therefore

$$\textbf{Ans.} = \frac{-3}{\pi} \arg(z^2 - 0) + \frac{-7}{\pi} \arg(z^2 - 1) + 7.$$
For the problem 9 it may be useful to recall that \((z + z^{-1})/2\) is a conformal map from \(\{\text{Im } z > 0, |z| > 1\}\) to the upper half space that leaves ±1 fixed and maps the intervals \(-\infty, -1[\) and \([1, \infty[\) to themselves.

9. (5+1 points) i. Define the argument to have values in \([-\pi, \pi[\) and suppose that \(0 < B < \pi\) Find a nonzero irrotational incompressible flow in the region

\[
\Omega := \left\{ z : 0 < \arg z < B, \quad 1 < |z| < \infty \right\},
\]

with flow parallel to the boundary at all boundary points.

ii. If \(G(z) = \phi + i\psi\) is the complex potential, it is true that one of \(\phi, \psi\) is constant on the boundary. Which one and why?

Solution. i. The map \(z \mapsto z^{\pi/B}\) maps the region to the part of the upper halfspace exterior to the unit disk.

The map \(z \mapsto (z + z^{-1})/2\) maps this region to the upper half space.

Therefore the map

\[
H(z) := \left(\frac{z^{\pi/B} + z^{-\pi/B}}{2}\right)
\]

maps the sector to the upper half space.

The function \(z\) is the complex potential of flow in the upper half space that it tangent to the boundaries. Therefore \(H(z)\) is the complex potential of a flow in \(\Omega\) that is tangent to the boundaries. If \(H = \phi + i\psi\) the velocity field of the irrotational incompressible flow is

\[
\nabla (\text{Re } H) = \nabla \phi.
\]

ii. The level sets of \(\psi\) are streamlines. So the boundary is level set of \(\psi\) when the boundary is connected.

Discussion. The velocity field constructed is not bounded. As in the Fluid Flow handout, one can show that there does not exist a bounded nonzero incompressible inrotational flow tangent to boundary.