1. If $F$ is a linear fractional transformation not equal to the identity show that there are at most two fixed points, that is values $z$ so that $F(z) = z$.

2. Suppose that $-\infty < a < b < \infty$ and denote by $u(x, y)$ the unique bounded harmonic function in $y > 0$ that attains value 0 on $] - \infty, a[$, 1 on $]a, b[$, and, 0 on $]b, \infty[$. Find a harmonic conjugate of $u$ in $y > 0$.

3. Find the unique steady state bounded temperature $u$ in $y > 0$ that on the $x$-axis has value 0 on $] - \infty, -1[$, 1 on $]1, \infty[$, while the interval $]-1, 1[$ is insulated.

4. A steel plate is formed by cutting the unit disk into two pieces and discarding the lower piece. The cut is along the horizontal line through the point $z = e^{i\theta}$ with $0 < \theta < \pi$. The circular boundary of the resulting plate is kept at temperature $T = 1$. The horizontal boundary at $T = 0$. Find the unique bounded steady state temperature distribution.

5. 245/18 Variant. Denote by $\Gamma$ the plane $\mathbb{R}^2$ with the segments $-\infty < x < -\sigma$ and $\sigma < x < \infty$ on the $x$-axis removed as in the figure. Find the unique temperature $u(x, y)$ harmonic and bounded on $\Gamma$ with $u = -1$ on the left hand removed half line and $u = 1$ on the right hand removed half line. **Hint.** We know how to find a bounded harmonic function in the upper half disk that vanishes on the diameter and is equal to one on the top of the disk.

Consider the domain $\Theta$ equal to the unit disk with center at the origin with the interval $-1 < x \leq 0$ on the $x$-axis removed. The map $z^{1/2}$ suitably defined maps $\Theta$ to the right half disk. This allows one to solve the Dirichlet problem on the slit disk $\Theta$ with value $-1$ on both sides of the slit and value 0 on the circumference of $\Theta$. Send the intersection of the slit with the circumference to infinity to solve a heat flow problem sketched in the third figure. Finish by using symmetry of the original problem with respect to the $y$-axis.
6. i. For \( \varepsilon > 0 \) denote by \( P_\pm := \pm \varepsilon \). Find the unique bounded harmonic function defined in the upper half plane \( y > 0 \), continuous on \( \{ \text{Im } z \geq 0 \} \setminus \{ P_\pm \} \) so that

\[
\begin{align*}
  u(x, 0) &= 0 \quad \text{when} \quad -\infty < x < -\varepsilon, \\
  u(x, 0) &= 1 \quad \text{when} \quad \varepsilon < x < \infty, \\
  \frac{\partial u(x, 0)}{\partial y} &= 0 \quad \text{when} \quad -\varepsilon < x < \varepsilon.
\end{align*}
\]

This is the steady state temperature distribution with semi-infinite rays at temperatures 0 and 1 separated by an insulated boundary strip of length \( 2\varepsilon \).

ii. Show that as \( \varepsilon \to 0 \) the solution tends to the corresponding problem without any insulating segment.

7. Find the unique bounded harmonic function in the upper half disk \( \{|z| < 1\} \cap \{ \text{Im } z > 0 \} \) that vanishes on the circular part of the boundary as well as on the boundary segment along the positive real axis and satisfies the Neumann boundary condition \( \partial u/\partial y = 0 \) on the boundary segment along the negative real axis.