Image of Spheres by Invertible $M$

Abstract This note presents a solution of the following problem which avoids conformal matrices and polar decomposition.

Problem. Show that the image of a sphere by an invertible linear transformation on $\mathbb{R}^N$ is an ellipsoid. Find an algorithm to compute the lengths and directions of the principal axes.

1 Scalar product.

Denote the standard scalar product of vectors in $\mathbb{R}^n$ by

$$\langle x, y \rangle = \sum x_i y_i.$$  

Vectors $x$ and $\bar{x}$ are orthogonal if and only if $\langle x, \bar{x} \rangle = 0$. An orthonormal basis for $\mathbb{R}^n$ is a set of mutually orthogonal unit vectors $e_1, \ldots, e_n$.

2 Transposes.

Suppose that $A_{ij}$ is an $n \times n$ real matrix. The transpose $A^t$ of $A$ is defined by

$$(A^t)_{ij} := A_{ji}.$$  

The matrix of the transpose is the matrix of $A$ flipped in the diagonal.

Example 2.1.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$  

Proposition 2.2. For all vectors $x$ and $y$, $n \times n$ matrices $A$ and $B$, and real numbers $\alpha$,

i. $(A + B)^t = A^t + B^t$,

ii. $(\alpha A)^t = \alpha A^t$,

iii. $(AB)^t = B^t A^t$,

iv. $\langle Ax, y \rangle = \langle x, A^t y \rangle$.

3 Positive symmetric matrices.

Definition 3.1. A symmetric real matrix $R$ is one for which $R_{ij} = R_{ji}$ for all $i, j$. Equivalently $R = R^t$.  

1
It is a fundamental fact that for every symmetric matrix there is an orthonormal basis \( e_1, e_2, \ldots, e_n \) for \( \mathbb{R}^n \) consisting of eigenvectors of \( R \),

\[
R e_j = \lambda_j e_j, \quad j = 1, \ldots, n.
\]

Expressing an arbitrary vector \( x \) in this basis yields

\[
x = \sum \alpha_j e_j, \quad \alpha_j = \langle x, e_j \rangle.
\]

Then,

\[
Rx = \sum \alpha_j R e_j = \sum \alpha_j \lambda_j e_j.
\]

The coordinate \( \alpha_j \) in the basis \( e_j \) is multiplied by \( \lambda_j \).

**Definition 3.2.** A symmetric real \( R \) is **positive** when all the eigenvalues are strictly positive.

**Proposition 3.3.** A symmetric matrix \( R \) is positive if and only if for all \( x \neq 0 \), \( \langle Rx, x \rangle > 0 \).

**Proof.** Compute in an orthonormal basis of eigenvectors,

\[
Rx = R(\sum \alpha_j e_j) = \sum \alpha_j Re_j = \sum \alpha_j \lambda_j e_j.
\]

Then compute

\[
\langle Rx, x \rangle = \left\langle \sum \alpha_j \lambda_j e_j, \sum \alpha_k e_k \right\rangle = \sum \alpha_j \alpha_k \lambda_j \langle e_j, e_k \rangle = \sum \lambda_j \alpha_j^2
\]

The proposition follows.

**Proposition 3.4.** If \( M \) is invertible then \( MM^t \) is a positive symmetric matrix.

**Proof.** Compute,

\[
(MM^t)^t = (M^t)^t M^t = MM^t,
\]

proving symmetry.

For \( x \neq 0 \),

\[
\langle M M^t x, x \rangle = \langle M^t x, M^t x \rangle = \|M^t x\|^2 > 0,
\]

proving positivity.
4 Image of a ball by an invertible $M$.

Theorem 4.1. If $M$ is an invertible linear transformation denote by $e_j$ and $\lambda_j > 0$ an orthonormal basis of eigenvectors of $MM^t$ and the corresponding eigenvalues. Then, the image of the unit sphere $\{\|x\| = 1\}$ by $M$ is the ellipsoid with principal axes of length $2/\sqrt{\lambda_j}$ along the directions $e_j$.

Proof. The image of the unit sphere is the set of vectors $y = Mx$ with $\|x\| = 1$. Write $x = M^{-1}y$. The $y$ are characterized by the equation $\|M^{-1}y\|^2 = 1$. Compute

$$\|M^{-1}y\|^2 = \langle M^{-1}y, M^{-1}y \rangle = \langle (M^{-1})^tM^{-1}y, y \rangle.$$ 

One has,

$$(M^{-1})^tM^{-1} = (MM^t)^{-1}.$$ 

Therefore, the vectors $e_j$ are eigenvectors of $(M^{-1})^tM^{-1}$ with eigenvalue $\lambda_j^{-1}$. In the basis $e_j$ with $y = \sum \beta_j e_j$ the image of the unit sphere is given by

$$\|M^{-1}y\|^2 = \sum \frac{\beta_j^2}{\lambda_j} = 1, \quad \sum \beta_j^2 = 1.$$ 

In the coordinates $\beta_j$ one has (by definition of ellipsoid) an ellipsoid with axes along the coordinate axes and of length $2/\sqrt{\lambda_j}$.

Since the coordinate axes in the $\beta$ coordinates are the directions $e_j$ in the original coordinates, this completes the proof.

Example. Compute the image of the unit circle by the Jacobian matrix

$$J = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$$

from class.

Solution. Compute

$$JJ^t = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}.$$ 

The eigenvalues are the roots $\lambda$ of,

$$0 = \det \begin{pmatrix} 5 - \lambda & 3 \\ 3 & 5 - \lambda \end{pmatrix} = (5 - \lambda)^2 - 9 = ((5 - \lambda) + 3)(5 - \lambda - 3)).$$
The roots are $\lambda = 5 \pm 3$. Eigenvalues and unit eigenvectors are

$$\lambda_1 = 2, \quad \mathbf{e}_1 = (1/\sqrt{2}, -1/\sqrt{2}), \quad \lambda_2 = 8, \quad \mathbf{e}_2 = (1/\sqrt{2}, 1/\sqrt{2}).$$

The image is the ellipse with major axis along the direction $(1, -1)$ with length $2/\sqrt{2}$ and minor axis along the direction $(1, 1)$ and length $2/\sqrt{8}$. 