Band Limited Signals

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In the first section we show that band limited signals are entire functions. This is then used to derive Shannon' Sampling Theorem and to prove the Fourier Series representation of periodic functions in the suceeding sections.

§1. The Paley Wiener Theorem.

The Fourier Integral Representation of a signal $f(t)$ is given by

$$f(t) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega\,, \tag{1}$$

where the Fourier Transform $\hat{f}$ is related to f by the nearly identical formula

$$\hat{f}(\omega) \;:=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt\,. \tag{2}$$

Using contour integration, we have verified this reciprocal relation in the two concrete cases of

$$f(t) = e^{-a|t|}, \quad a > 0\,, \qquad \hat{f}(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+\xi^2}\,, \tag{3}$$

and

$$f(t) = e^{-t^2/2}\,, \qquad \hat{f}(\omega) = e^{-\xi^2/2}\,. \tag{4}$$

Once the relation (1), (2) is proved for a single positive signal f it is not difficult to prove that it is true in great generality.

A signal is called **band limited** if there is an $\Omega > 0$ so that $f(\omega) = 0$ for all $|\omega| > \Omega$. Such a signal is then given by

$$f(t) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \hat{f}(\omega)\, e^{i\omega t}\, d\omega\,, \tag{5}$$

We suppose that $\hat{f}(\omega)$ in an absolutely integrable function.

The formula on the right hand side of (5) is defined for all complex numbers t. This defines an extension of f to a function on the whole complex plane $\mathbb{C}$. The next result is half of a Theorem of Paley and Wiener. The other half of the Theorem concerns the converse.

1/2 Paley-Wiener Theorem. *Every band limited signal f is entire analytic function of $t \in \mathbb{C}$. If its spectrum is contained in $[-\Omega, \Omega]$ as in (5) and $\hat{f}$ is an integrable function then f is of exponential growth in the sense that for all $t \in \mathbb{C}$*

$$|f(t)| \;\le\; e^{\Omega|\mathrm{Im}\, t|} \int_{-\Omega}^{\Omega} |\hat{f}(\omega)|\, \frac{d\omega}{\sqrt{2\pi}}\,. \tag{6}$$

Proof. Write $t = \xi + i\eta$ in terms of its real an imaginary parts. Differentiating under the integral sign shows that f has partial derivatives of all order with respect to ξ and η and

$$\frac{\partial f}{\partial \xi} = \int_{-\Omega}^{\Omega} f(\omega)\, \frac{\partial e^{-it\omega}}{\partial \xi}\, d\omega\,, \qquad \frac{\partial f}{\partial \eta} = \int_{-\Omega}^{\Omega} f(\omega)\, \frac{\partial e^{-it\omega}}{\partial \eta}\, d\omega\,.$$

Since $e^{-it\omega}$ is an analytic function of t it satisfies the Cauchy-Riemann equation

$$\frac{\partial e^{-it\omega}}{\partial \xi} = \frac{1}{i}\,\frac{\partial e^{-it\omega}}{\partial \eta}\,.$$

Therefore f also satisfies the Cauchy-Riemann equations, so f is an entire analytic function. Since $e^{-it\omega} = e^{-it\xi}e^{\eta\omega}$, it follows that for $\omega \in [-\Omega, \Omega]$,

$$|e^{-it\omega}| \le e^{\Omega|\mathrm{Im}\,\zeta|}\,, \qquad \text{so,} \qquad |f(t)| \le \frac{e^{\Omega|\mathrm{Im}\,t|}}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} |\hat{f}(\omega)|\, d\omega\,.$$

The proof is complete. ∎

The analyticity of band limited signals has striking consequences. One is the unintuitive result that knowlege of the signal $f(t)$ on any arbitrarily short interval of time $a < t < b$ on the real axis determines its values at all times. This is a straightforward consequence of the unique continuation principal for analytic functions. If there were a second signal $\tilde{f}$ which agreed with f for $a < t < b$ then the difference $f - \tilde{f}$ would be entire and would vanish on $a < t < b$ and therefore would vanish identically.

Discussion. It is standard engineering wisdom that one cannot generate waves of arbitrarily short wavelength so that all signals are band limited. It is also standard wisdom that no signals extend infinitely far into the past and future. That is there is a $T > 0$ so that $f(t) = 0$ whenever t is real and $|t| > T$. These two together imply that all signals are entire analytic functions which vanish on $]-\infty, -T] \subset \mathbb{R}$. The unique continuation principal for analytic functions implies that all such signals must vanish identically. Thus the only signal satisfying the two conditions of engineering wisdom is the identically vanishing signal! The resolution of this paradoxical result is that neither the limit to the band $-\Omega < \omega < \Omega$ nor to the time interval $-T < t < T$ is exact.

In addition to the above stiking paradox we will give two applications of the Paley-Wiener Theorem. The first is to the fundamental Sampling Theorem from signal processing and the second is a complex variables derivation of the Fourier Series representation of periodic functions. For the latter we will need a slight strengthening of the inequality. Those only interested in the Sampling Theorem should skip directly to the next section.

Corollary. *With f as in the Paley-Wiener Theorem,*

$$\lim_{|\eta|\to\infty} \frac{|f(\xi + i\eta)|}{e^{\Omega|\eta|}} = 0 \quad \text{uniformly in } \xi\,.$$

Proof. For $0 < \delta << \Omega$ break the representation (5) into two pieces

$$f(t) \;=\; \frac{1}{\sqrt{2\pi}} \int_{|\omega|<\Omega-\delta} \hat{f}(\omega)\; e^{i\omega t}\; d\omega \;+\; \frac{1}{\sqrt{2\pi}} \int_{\Omega-\delta<|\omega|<\Omega} \hat{f}(\omega)\; e^{i\omega t}\; d\omega\,,$$

For the first use (6) with band width equal to $\Omega - \delta$ and for the second use (6) to obtain with a constant independent of δ and $\mathrm{Re}\, t$,

$$|f(t)| \;\le\; C\, e^{|\mathrm{Im}\, t|(\Omega-\delta)} + C\, e^{|\mathrm{Im}\, t|\Omega} \int_{\Omega-\delta<|\omega|<\Omega} |\hat{f}(\omega)|\; d\omega\,.$$

Since

$$\int_{\Omega-\delta<|\omega|<\Omega} |\hat{f}(\omega)|\; d\omega \;=\; o(1) \quad \text{as} \quad \delta \to 0\,,$$

the result follows. ∎

§2. Shannon's Sampling Theorem.

A consequence of analyticity which has great practical utility is that a band limited signal $f(t)$ can be recovered from its regularly spaced values

$$f(nL)\,, \qquad n \in \mathbb{Z}$$

providing that the spacing L of the sampling is sufficiently small. The hypothesis of the theorem is motivated by the example of the signal

$$f(t) = \sin \Lambda t = \frac{e^{i\Omega t} - e^{-i\Omega t}}{2\pi}$$

which is a limit of band limited signals with spectrum concentrated near the two points $\pm\Omega$. The limiting signal vanishes at the points $n\pi/\Omega$ with spacing π/Ω.

Shannon's Sampling Theorem. *If f is a band limited signal with band width Ω and $L < \pi/\Omega$ then f is reconstructed from its values sampled at the times $\{nL \,:\, xn \in \mathbb{Z}\}$ as the sum of the convergent series*

$$f(t) \;=\; \sum_{n=-\infty}^{n=\infty} \frac{(-1)^{n+1} L\, \sin(\pi t/L)}{\pi\,(nL - t)}\; f(nL)\,.$$

Remark. For $t = mL$, the summands on the right with $n \neq m$ all vanish. The summand with $n = m$ is defined by setting

$$\frac{L\, \sin(\pi t/L)}{\pi\,(nL - t)}\bigg|_{t=nL} \;=\; -1\,.$$

With that natural definition, the sampling identity is trivially satisfied for $t \in L\mathbb{Z}$.

Proof. For $t \notin \text{Ł}\mathbb{Z}$ consider the function

$$g(z) = \frac{f(z)}{(z-t)\sin(\pi z/L)}\,.$$

The Paley-Wiener Theorem shows that f is entire and it follows that g is analytic at all points of $\mathbb{C}$ except t and the roots, nL, of $\sin(\pi z/L)$.

At those roots one has

$$\frac{d\sin(\pi t/L)}{dt}\bigg|_{t=nL} \;=\; \frac{\pi}{L}\cos(\pi n) \;=\; \frac{(-1)^n\pi}{L} \neq 0.\,,$$

Therefore the roots are simple, g has at worst a simple pole, and

$$\text{Res}(g, nL) \;=\; \frac{(-1)^n\, L\, f(nL)}{\pi\,(nL-t)}\,. \tag{7}$$

The function $g(z)$ also a simple pole at t with

$$\text{Res}(g, t) \;=\; \frac{f(t)}{\sin(\pi t/L)}\,. \tag{8}$$

For positive integers N_1, N_2 and M, define a rectanglex

$$R_{N_1,N_2,M} \;:=\; \Big\{ z\in\mathbb{C} \;:\; -N_1 - \frac{L}{2} < \text{Re}\, z < N_2 + \frac{L}{2} \quad \text{and} \quad |\text{Im}\, z| < M \Big\}\,.$$

When N_1, N_2 and M are larger than $|t|$, the boundary does not hit any of the singularities of g so the Residue Theorem implies that for $t \notin L\mathbb{Z}$

$$\frac{1}{2\pi i}\oint_{\partial R_{N_1,N_2,M}} g(\tau)\; d\tau \;=\; \text{Res}(g,t) \;+\; \sum_{n=-N}^{n=N} \text{Res}\,(g, nL) \;=\; \frac{f(t)}{\sin(\pi t/L)} \;+\; \sum_{n=-N_1}^{n=N_2} \frac{(-1)^n L\, f(nL)}{\pi\,(nL-t)}\,.$$

On $\partial R_{N_1,N_2,M}$ one has with constants indpendent of N_1, N_2, M but depending on t,

$$\left|\frac{1}{\sin(\pi z/L)}\right| \le Ce^{-\pi|\text{Im}\, z|/L}, \qquad |f(z)| \le Ce^{|\text{Im}\, z|\Omega}, \qquad \left|\frac{1}{z-t}\right| \le \frac{C}{\min\{N_1, N_2, M\}}\,. \tag{9}$$

Fix t and $N_1, N_2 > |t|$ and let $M \to \infty$. The horizontal sides of $R_{N_1,N_2,M}$ have finite length and the integrand tends uniformly to zero since the decay of $1/\sin(\pi z/L)$ beats the growth of f because of the hypothesis $\pi/L > \Omega$. Thus

$$\frac{1}{2\pi i}\int_{\text{Re}\,\tau = N_2 + L/2} g(\tau)\, d\tau - \frac{1}{2\pi i}\int_{\text{Re}\,\tau=-N_1-L/2} g(\tau)\, d\tau \;=\; \frac{f(t)}{\sin(\pi t/L)} \;+\; \sum_{n=-N_1}^{n=N_2} \frac{(-1)^n L\, f(nL)}{\pi\,(nL-t)}\,.$$

To complete the proof of the Theorem it suffices to show that each of the integrals on the left tend to zero as $N_1, N_2 \to \infty$. From (9), the absolute value of the integrand is bounded above by

$$\frac{C}{\min\{N_1, N_2\}}\; e^{-\pi|\text{Im}\,\tau|(\frac{\pi}{L}-\Omega)}\,.$$

Since $\frac{\pi}{L} - \Omega > 0$, the integrals are $O(1/\min\{N_1, N_2\})$ completing the proof. ∎

§3. A complex variable proof of convergence of Fourier Series.

Using Laurent expansions, we have shown that every 2π periodic function which is analytic in a neighborhood of the real axis has a Fourier series representation

$$f(\theta) = \sum_{n\in\mathbb{Z}} a_n\, e^{in\theta}\,, \tag{1}$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta\,. \tag{2}$$

This representation is valid in great generality. For square integrable f, the convergence is in the L^2 norm, for periodic distributions in the sense of Schwartz the convergence is in the sense of distributions, and for infinitely differentiable periodic f the convergence of each differentiated series is uniform. The C^∞ result implies the previous two (see remarks below). In this note, we show how that result can be proved using only results of elementary complex analysis. We use neither approximate delta functions nor the Weierstrass approximation theorem (see Exercise 2.).

Theorem. *If $f \in C^\infty(\mathbb{R})$ is 2π periodic then the representation (1), (2) is valid. The Fourier coefficients (2) satisfy the rapid decay estimate*

$$\forall\, N,\ \exists\, C_N,\ \forall\, n,\quad |a_n| \ \le\ \frac{C_N}{(1+|n|)^N}\,, \tag{3}$$

so the series and all differentiated series converge uniformly on $\mathbb{R}$.

Proof. Integration by parts with an induction on N shows that

$$(-in)^N a_n \ =\ \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\ \frac{d^N}{d\theta^N} f(\theta)\ d\theta$$

which implies (3).

It follows that the Fourier series on the right of (1) converges uniformly with all of its derivatives to an infinitely differentiable 2π periodic function $\tilde f$. Passing the integral through the sum shows that the Fourier coefficients of $\tilde f$ are equal to the Fourier coefficients of f. Thus $g := f - \tilde f$ is an infinitely differentiable smooth periodic function all of whose Fourier coefficients vanish. It suffices to show that g is identically equal to 0.

For $\zeta \in \mathbb{C}$, introduce the band limited Fourier Transform

$$F(\zeta)\ :=\ \int_{-\pi}^{\pi} g(\theta)\, e^{-i\theta\zeta}\, d\theta\,, \qquad \zeta = \xi + i\eta\,, \quad \xi, \eta \in \mathbb{R}\,. \tag{4}$$

The vanishing of the Fourier coefficients yields

$$F(n)\ =\ 0\,, \qquad \text{for all}\quad n \in \mathbb{Z}\,. \tag{5}$$

The Paley-Wiener Theorem implies that F is an entire analytic function.

Consider the quotient

$$G(\zeta)\ :=\ \frac{F(\zeta)}{\sin \pi\zeta}\,. \tag{6}$$

The function G is analytic except possibly for isolated singularities at the zeroes $\zeta = n \in \mathbb{Z}$ of $\sin \pi\zeta$. These are simple zeroes of the sine function and F vanishes at these points by (5). Consequently, G has a removable singularity at each of these points (Exercise 1.) Therefore, G is an entire analytic function.

For positive integers M and N define the rectangle

$$R_{M,N} \ := \ \Big\{ \zeta \ : \ |\mathrm{Re}\, \zeta| \le N + \frac{1}{2} \ \text{ and } \ |\mathrm{Im}\, \zeta| \le M \Big\}.$$

The estimate from the Paley Wiener Theorem shows that on the vertical sides of the rectangle one has

$$|G(\zeta)| \ \le \ \sup_{\eta \in \mathbb{R}} \ \frac{e^{\pi|\eta|}}{2\pi \ |\sin \pi(\frac{1}{2} + i\eta)|} \ \int_{-\pi}^{\pi} |g(\theta)| \ d\theta \, . \ := \ C_1 < \infty \, .$$

The estimate from the Corollary to the Paley Wiener Theorem shows that as $M \to \infty$, G converges uniformly to zero on the horizontal sides of $R_{M,N}$. The maximum modulus principal when M is large shows that $|G| \le C_1$ on $R_{M,N}$. Letting $M \to \infty$ and then $N \to \infty$ shows that $|G| \le C_1$ everywhere.

Liouville's Theorem implies that G is constant. The Corollary to the Paley Wiener Theorem shows that G tends to zero as $|\mathrm{Im}\, \zeta| \to \infty$. Thus the constant value must be zero, so, G is identically equal to zero.

Then $F(\zeta) = G(\zeta) \ \sin \pi\zeta = 0$.

For any $\underline{\theta} \in [-\pi, \pi]$, define band limited signals

$$F_+(\zeta) := \int_{-\pi}^{\underline{\theta}} e^{-i(\theta - \underline{\theta})\zeta} \ g(\theta) \ d\theta \, , \qquad F_-(\zeta) := \int_{\underline{\theta}}^{\pi} e^{-i(\theta - \underline{\theta})\zeta} \ g(\theta) \ d\theta \, .$$

Then $F_+ + F_- = e^{i\underline{\theta}\zeta} F = 0$.

Both $F_\pm$ are entire analytic functions. The function F_+ is uniformly bounded in the lower half plane $\mathrm{Im}\, \zeta \le 0$ while F_- is bounded in $\mathrm{Im}\, \zeta \ge 0$. Thus the function defined as F_+ in the lower halfplane and $-F_-$ in the upper is entire and bounded. Liouville's theorem implies that this function must be constant and therefore F_+ is constant.

As in the demonstration of the Corollary to the Paley Wiener Theorem, one has $\lim F_+(\xi + i\eta) \to 0$ as $\eta \to -\infty$ so the constant value must equal 0. Taking $\zeta = 0$ in the definition of F_+ one has

$$\int_{-\pi}^{\underline{\theta}} g(\theta) \ d\theta = F_+(0) = 0 \, .$$

This is true for all $\underline{\theta}$, so, the integral of g over every interval vanishes. This implies that $g = 0$. ∎

Remarks. 1. This elegant argument proves the Theorem without explaining why it is true.

2. The proof shows that a Lebesgue integrable periodic function g with vanishing Fourier coefficients must vanish.

3. Since square integrable functions are integrable, Remark 2 suffices to establish that $\{e^{in\theta}/\sqrt{2\pi}\}$ is a complete orthonormal family in the square integrable periodic functions.

4. Convergence of the Fourier expansion of periodic distributions follows from the Theorem by a duality argument.

5. The division by $\sin \pi\zeta$ recalls problem 197/8 in Churchill-Brown and the proof of the Sampling Theorem.

Exercises.

1. Give details of the argument showing that G has removable singularities at $\zeta = n\pi$ with $n \in \mathbb{Z}$.

2. The Weierstrass Approximation Theorem asserts that if f is a continuous function on an interval, then on that interval f can be uniformly approximated by polynomials. Prove this as follows. Show that it suffices to consider the interval $I = [-1, 1]$. Given a continuous function on $[-1, 1]$, show that there is a continuous 2π periodic extension to all of $\mathbb{R}$. Show that the periodic extension is the uniform limit of infinitely smooth 2π periodic functions. Hint: Convolution with an approximate delta. Then approximate the smooth periodic function with a trigonometric polynomial by truncating the Fourier representation. Then approximate by a polynomial by approximating each exponential by a Taylor polynomial.