Fourier Series, Integrals, and, Sampling From Basic Complex Analysis

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Outline. The Fourier series representation of analytic functions is derived from Laurent expansions. Elementary complex analysis is used to derive additional fundamental results in harmonic analysis including the representation of C^∞ periodic functions by Fourier series, the representation of rapidly decreasing functions by Fourier integrals, and Shannon's sampling theorem. The ideas are classical and of transcendent beauty.

§1. Laurent series yield Fourier series.

A difficult thing to understand and/or motivate is the fact that arbitrary periodic functions have Fourier series representations. In this section we prove that periodic analytic functions have such a representation using Laurent expansions.

Definition. *A function $f(z)$ defined on a strip*

$$\{z \ : \ |\mathrm{Im}\, z| < a\}\,, \qquad a > 0\,, \tag{1.1}$$

is 2π **periodic** *if for all such z,*

$$f(z + 2\pi) = f(z)\,. \tag{1.2}$$

Examples of periodic analytic functions. *The elementary functions $\sin nz$, $\cos nz$, and $e^{\pm inz}$ are the building blocks. Any finite linear combination is an example. Nonlinear functions too, for example*

$$\frac{1}{1+\sin^2 z}$$

is analytic in any strip on which $\sin z \neq \pm i$. An entire function $h = \sum_0^\infty a_n\, z^n$ yields the entire example

$$h(e^{iz}) \ = \ \sum_0^\infty a_n\, e^{inz}\,.$$

An analysis related to the last example yields the general case. Consider the mapping

$$z \ \mapsto \ w \ = \ e^{iz}\,. \tag{1.3}$$

It maps the strip (1.1) onto the annulus

$$\{w \ : \ e^{-a} < |w| < e^{a}\}\,. \tag{1.4}$$

It maps the real axis infinitely often around the unit circle in the w plane. The preimages of a point $w = e^{i\theta}$ are the points $z = \theta + 2\pi n$ with $n \in \mathbb{Z}$. Since the derivative dw/dz is nowhere zero, the mapping is locally invertible with analytic inverse. The local inverses are branches of the function $z = (\ln w)/i$.

Theorem 1.1. *The correspondence $g(w) \mapsto f(z)$,*

$$f(z) \ = \ g(e^{iz}) \tag{1.5}$$

establishes a one to one correspondence between the analytic functions $g(w)$ on the annulus (1.4) the 2π periodic analytic functions $f(z)$ in the strip (1.1).

Proof. That each such g yields an analytic periodic f on the strip and that distinct functions g yield distinct f is clear. It suffices to show that every f has such a representation.

Suppose that f is analytic and periodic in the strip. For each point w in the annulus, the preimages z lie in the strip and differ by integer multiples of 2π. Thus, the function f has the same value at all the preimages. It follows that a function g on the annulus is well defined by the formula $g(w) = f(z)$ since it does not matter which z one takes.

For any $\underline{w}$ choose a preimage $\underline{z}$. The Inverse Function Theorem implies that w has a local inverse $z = F(w)$ analytic on a neighborhood of $\underline{w}$ and satisfying $F(\underline{w}) = \underline{z}$. Near $\underline{w}$, $g(w) = f(F(w))$ is therefore analytic. Thus g provides the desired representation of f. ∎

Theorem 1.2. *If $f(z)$ is a 2π periodic analytic function in the strip (1.2) then f has a Fourier series representation*

$$f(z) \;=\; \sum_{n=-\infty}^{\infty} c_n\, e^{inz}\,, \tag{1.6}$$

uniformly convergent on each thinner strip. The coefficients are given by the formulas

$$c_n \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, e^{-in\theta}\, d\theta\,. \tag{1.7}$$

Proof. Choose g so that (1.5) holds. Then use the Laurent expansion of g

$$g(w) \;=\; \sum_{-\infty}^{\infty} c_n\, w^n\,, \qquad c_n \;=\; \frac{1}{2\pi i}\oint_{|w|=1} \frac{g(w)}{w^{n+1}}\, dw\,, \tag{1.8}$$

uniformly convergent on each subannulus.

Since $f(z) = g(e^{iz})$, one has

$$f(z) \;=\; \sum_{-\infty}^{\infty} c_n\, (e^{iz})^n$$

which is formula (1.6).

Parameterizing the curve $|w| = 1$ by $w = e^{i\theta}$ with $0 \le \theta \le 2\pi$, one has $dw = i\, w\, d\theta$ and the formula for c_n becomes

$$c_n \;=\; \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(e^{i\theta})}{w^{n+1}}\, i\, w\, d\theta \;=\; \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)}{w^n}\, d\theta\,,$$

which proves (1.7). ∎

Fourier series were discovered before Laurent expansions. If history were more logical they might have been found this way.

§2. Paley-Weiner for Fourier series.

Every 2π periodic function that is analytic in a neighborhood of the real axis has a Fourier series representation (1.6)-(1.7). If (1.6) holds, multiplying by e^{-imx} then integrating over $[-\pi, \pi]$ yields the formula (1.6) for the coefficients since

$$\int_{-\pi}^{\pi} e^{-imx}\, e^{inx}\, dx \;=\; \begin{cases} 2\pi & \text{when} \quad m = n \\ 0 & \text{when} \quad m \ne n\,. \end{cases}$$

Periodic functions that need not be analytic have Fourier expansion of the same form. The smoother are the functions, the more rapidly decreasing are the coefficients c_n and the faster is the convergence in (1.6). Analytic periodic f are characterized by the fact that the c_n are exponentially decreasing as $|n| \to \infty$.

Paley-Weiner Theorem 2.1. i. *If f is an analytic periodic function in the strip (1.1), then its Fourier coefficients c_n satisfty for any $\epsilon > 0$ there is $C(\epsilon)$ so that*

$$|c_n| \;\leq\; C(\epsilon)\, e^{(-a+\epsilon)|n|}\,. \tag{2.1}$$

ii. *Conversely, if f is given by (1.6) with c_n satisfying (2.1) then f has an analytic continuation to the strip (1.1).*

Exercise 2.1. *Prove* **i. Hint.** For $n > 0$ start from the formula for c_n in (1.8). For $1 < b < e^a$ move the contour to $|z| = b$ using Cauchy's Theorem. On that contour, $1/w^{n+1}$ is exponentially small as $n \to \infty$. Perform an analogous estimate to treat $n < 0$. Alternatively use the fact that the Laurent expansion is convergent to estimate the Laurent coefficients.

Exercise 2.2. *Prove* **ii. Hint.** The Fourier series is uniformly convergent on any strip thinner than (1.1)

§3. Fourier series for nonanalytic periodic functions.

For infinitely differentiable periodic f the c_n decrease faster than any negative power of $|n|$, that is,

$$\forall\, N,\; \exists\, C_N,\; \forall\, n, \quad |c_n| \;\leq\; \frac{C_N}{\langle n\rangle^N}\,, \qquad \langle n\rangle := (1+|n|^2)^{1/2}\,. \tag{3.1}$$

This is a consequence of the formula for the Fourier coefficients of the derivative,

$$c_n(f') \;=\; in\, c_n(f)\,, \tag{3.2}$$

valid for example if f is continuously differentiable. The proof of (3.2) is by integrating by parts with boundary terms cancelling by periodicity to give,

$$2\pi\, c_n(f') \;=\; \int_{-\pi}^{\pi} f'(x)\; e^{-inx}\; dx \;=\; -\int_{-\pi}^{\pi} f(x)\; \frac{de^{-inx}}{dx}\; dx \;=\; 2\pi\, in\, c_n(f)\,. \tag{2.3}$$

Therefore if $f \in C^N$,

$$\big|(-in)^N c_n(f)\big| \;=\; \big|c_n(f^{(N)})\big| \;\leq\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|\frac{d^N f}{dx^N}(x)\Big|\; dx \tag{2.4}$$

which implies (2.1).

For smooth periodic f the Fourier series and each differentiated series converges uniformly. For less regular f the convergence is less strong. For example, for f which are merely square integrable, one has only $\sum |c_n|^2 \;<\; \infty$, the convergence is in the root mean square sense. For periodic distributions in the sense of Schwartz, the convergence is in the sense of distributions. The C^∞ result implies the others (see remarks below). We prove the C^∞ result using complex analysis.

Theorem 3.1. *If f is an infinitely differentiable 2π periodic function on the real line, then the representation (1.6)–(1.7) is valid. The Fourier coefficients (1.7) satisfy the rapid decay estimate (3.1) so the series and all differentiated series converge uniformly on $\mathbb{R}$.*

Historically, many examples of expansions (1.6) were discovered before it was realized how general was the phenomenon. For example if $|a| < 1$, one has

$$\frac{1}{1 - a\sin\theta} \;=\; \sum_{n=0}^{\infty} \big(a\sin\theta\big)^n .$$

It was Fourier who uncovered the fact that the representations were general and their utility in analysing differential equations. This preceded the flowering of complex analysis.

§4. The Fourier transform.

Our treatment of Fourier series is intimately entangled with the Fourier transform representation

$$g(x) \;=\; \int_{-\infty}^{\infty} \hat g(\xi)\, e^{ix\xi}\, d\xi\,, \tag{4.1}$$

$$\hat g(\xi) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat g(x)\, e^{-ix\xi}\, dx \;:=\; \mathcal{F}(g)(\xi)\,, \tag{4.2}$$

for functions g defined on $\mathbb{R}$ so that g and $\hat g$ tending to zero sufficiently fast at $\pm\infty$. Using contour integration, this reciprocal relation is verified in the two concrete cases,

$$g(x) = e^{-a|x|}, \quad a>0\,, \qquad \hat g(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+\xi^2}\,, \tag{4.3}$$

and

$$g(x) = e^{-x^2/2}\,, \qquad \hat g(\omega) = e^{-\xi^2/2}\,. \tag{4.4}$$

The standard proofs of (4.1), (4.2) rely on one of these two examples.

A convenient class of functions for studying the Fourier transform is the Schwartz class $\mathcal{S}$ consisting of those g so that for all $0 < n, m$ there is a $C(n,m)$ so that

$$\left|\frac{d^m g}{dx^m}\right| \;\le\; C\,\langle x\rangle^{-n}\,.$$

For such g, an integration by parts as in (3.3) shows that

$$\mathcal{F}(g') \;=\; i\xi\,\hat g\,. \tag{4.5}$$

Differentiating the definition of $\hat g$ yields

$$\frac{d}{d\xi}\hat g = \mathcal{F}(-ix\,g)\,. \tag{4.6}$$

It follows that the Fourier transform of a function in $\mathcal{S}$ belongs to $\mathcal{S}$ so that in (4.1), (4.2) the integrals are very rapidly convergent.

Exercise 4.1. *Define the inverse transform $h(\xi) \mapsto \mathcal{F}^* h$ by*

$$\mathcal{F}^* h(x) \;:=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\, h(\xi)\, dx\,.$$

Derive formulas analogous to the preceding two for $\mathcal{F}^* h'$ and $\mathcal{F}^*(\xi h)$. From those prove that the operator $\mathcal{F}^*\mathcal{F}$ from $\mathcal{S}$ to itself commutes with multiplicalion by x and also with d/dx.

The analysis requires the following fundamental result.

Riemann-Lebesgue Lemma 4.1. *If $g(x)$ is an absolutely integrable function on $\mathbb{R}$, then*

$$\lim_{|\xi|\to\infty} \hat{g}(\xi) \;=\; 0\,.$$

Proof. For $\epsilon > 0$, choose $\psi \in \mathcal{S}$ so that

$$\int_{-\infty}^{\infty} \big|g(x) - \psi(x)\big|\; dx \;<\; \epsilon\,.$$

Then

$$\forall \xi \in \mathbb{R}\,, \quad \big|\hat{g}(\xi) \;-\hat{\psi}(\xi)\big| \;<\; \frac{\epsilon}{2\pi}\,.$$

Since $\hat{\psi}(\xi) \to 0$ as $|\xi| \to \infty$, one has

$$\limsup_{|\xi|\to\infty}\; |\hat{g}(\xi)| \;\leq\; \frac{\epsilon}{2\pi}\,.$$

Since this is true for any $\epsilon > 0$, the proof is complete. ∎

Example. This result applied to $g = \chi_{[-\pi,\pi]}(x)\, f(x)$ with g a 2π-periodic function implies that the Fourier coefficients of an periodic function tend to zero as $n \to \infty$.

§5. Uniqueness of Fourier transforms, proof of Theorem 3.1.

The key step in the proof of (1.6), (1.7) is to prove that if a periodic function f has all its Fourier coefficients equal to zero, then the function vanishes. Similarly if an absolutely integrable function g on $\mathbb{R}$, has Fourier transform $\hat{g}$ identically equal to 0, then $g = 0$. Equivalently, if two periodic functions f_1 and f_2 have the same Fourier coefficients, then $f_1 = f_2$, and if g_1 and g_2 are absolutely integrable functions on $\mathbb{R}$ that have the same Fourier transforms, then $g_1 = g_2$. These equivalences follow from applying the preceding assertions to $f := f_1 - f_2$ and $g := g_1 - g_2$ respectively.

Theorem 5.1. 1. *If $g(x)$ is an absolutely integrable function on $\mathbb{R}$ whose Fourier transform is identically equal to zero, then $g = 0$.*

2. *If $f(x)$ is a 2π periodic function absolutely integrable over each period whose Fourier coefficients are all equal to zero, then $f = 0$.*

Proof. 1. Write (ingeneously!)

$$\hat{g} = F_- + F_+\,, \qquad F_-(\zeta) := \int_{-\infty}^{0} g(x)\, e^{-ix\zeta}\, dx\,, \quad F_+(\zeta) := \int_{0}^{\infty} g(x)\, e^{-i\zeta x}\, dx\,.$$

The first observation is a Paley-Weiner result. The function F_+ has an analytic continuation into the lower half plane and F_- into the upper. Write $\zeta = \xi + i\eta$ in terms of its real and imaginary parts. Then

$$e^{-i\zeta x} = e^{-i\xi x}\, e^{\eta x}\,.$$

For $x \geq 0$, $e^{i\zeta x}$ is uniformly bounded in $\mathrm{Im}\,\zeta \leq 0$ and for $\mathrm{Im}\,\zeta < 0$ decays exponentially as $x \to \infty$. The functions F_+, are continuous and uniformly bounded on $\{\mathrm{Im}\,\zeta \leq 0\}$. Differentiating under the integral shows that F_+ is analytic in the interior, $\{\mathrm{Im}\,\zeta < 0\}$. Similarly F_- is analytic in $\{\mathrm{Im}\,\zeta > 0\}$ and continuous and uniformly bounded in $\{\mathrm{Im}\,\zeta \geq 0\}$.

On the boundary of the two half spaces one has $F_+ + F_- = \hat{g} = 0$. Therefore the function

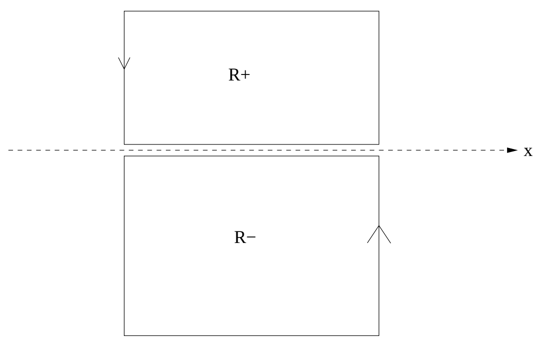
$$H(\zeta) \;:=\; \pm F_\pm \quad \text{when} \quad \mp\,\mathrm{Im}\,\zeta \geq 0\,,$$

is holomorphic and uniformly bounded on each half space. The relation $F_+ = -F_-$ on the real axis establishes the continuity of H at those points.

Cauchy's Theorem implies that

$$\oint_{\partial R} H(z)\ dz \;=\; 0$$

for all rectangles contained in either the upper or lower half planes. If a rectangle crosses the x-axis as in the figure



denote by $R_\pm$ the intersection with $\pm\mathrm{Im}\,z > 0\}$. Since the integrals over the bounding edges on the x axis cancel exactly one has

$$\oint_{\partial R} H(z)\ dz \;=\; \oint_{\partial R_+} H(z)\ dz \;+\; \oint_{\partial R_-} H(z)\ dz \;=\; 0 \;+\; 0 \;=\; 0\,.$$

Morera's theorem implies that H is an entire analytic function by constructing an antiderivative by integration along arcs consisting of a finite number of horizontal and vertical segments. Therefore H is a bounded entire function. Liouville's Theorem implies that H is constant.

The Riemann-Lebesgue Lemma implies that H tends to zero at infinity on the real axis, so the constant must be 0. Therefore $H = 0$. In particular,

$$0 = H(0) = F_+(0) = \int_{-\infty}^{0} g(\theta)\, d\theta\,.$$

For any $a \in \mathbb{R}$, the function $g(x-a)$ also has vanishing Fourier Transform since

$$\int g(x-a)\ e^{-ix\xi}\ dx \;=\; \int f(x)\ e^{-i(x+a)\xi}\ dx \;=\; e^{-ia\xi} \int g(x)\ e^{-ix\xi}\ dx \;=\; 0\,.$$

Therefore

$$0 \;=\; \int_{-\infty}^{0} g(x-a)\ dx \;=\; \int_{-\infty}^{-a} g(x)\ dx\,.$$

Since this is true for all a it follows that $g = 0$.

2. For $\zeta \in \mathbb{C}$, introduce the Fourier Transform

$$F(\zeta) \ := \ \int_{-\pi}^{\pi} f(\theta)\, e^{-i\theta\zeta}\, d\theta\,, \qquad \zeta = \xi + i\eta\,, \quad \xi, \eta \in \mathbb{R}\,. \tag{5.1}$$

The vanishing of the Fourier coefficients yields

$$F(n) \ = \ 0\,, \qquad \text{for all} \quad n \in \mathbb{Z}\,. \tag{5.2}$$

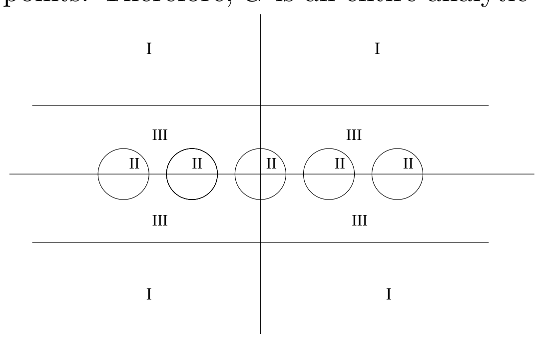
Differentiating under the integral sign shows that F is an entire analytic function of ζ.
Since $e^{-i\theta\zeta} = e^{-i\theta\xi}e^{\eta\theta}$, it follows that for $\theta \in [-\pi, \pi]$,

$$|e^{-i\theta\zeta}| \ \le \ e^{\pi|\mathrm{Im}\,\zeta|}\,, \qquad \text{so,} \qquad |F(\zeta)| \ \le \ e^{\pi|\mathrm{Im}\,\zeta|} \int_{-\pi}^{\pi} |g(\theta)|\, d\theta\,. \tag{5.3}$$

The strategy is to prove that $F = 0$. Then part **1** implies that $f\chi_{[-\pi,\pi]} = 0$ and therefore that $f = 0$. Define

$$G(\zeta) \ := \ \frac{F(\zeta)}{\sin \pi\zeta}\,.$$

G is analytic except possibly for isolated singularities at the zeroes $\zeta = n \in \mathbb{Z}$ of $\sin \pi\zeta$. These are simple zeroes of $\sin z$ and F vanishes at these points by (5.2). Consequently, G has a removable singularity at each of these points. Therefore, G is an entire analytic function.



For $|\mathrm{Im}\,\zeta| \ge 1$ (region I in the figure), there is a $C > 0$, so that $|\sin \pi\zeta\,| \ge C\, e^{\pi|\mathrm{Im}\,\zeta|}$. Therefore (5.3) shows that G is uniformly bounded on region I.

Fix $1 > \rho > 0$ and consider the region III which consists of the points in $|\mathrm{Im}\,\zeta| \le 1$ and outside the union of disks of radius ρ and centers n. Both $F(\zeta)$ and $1/\sin \pi\zeta$ are bounded in III, so G is bounded there.

Since F is bounded on $\mathrm{Im}\,\zeta \ \le \ 1$ Cauchy's inequalities imply that F' is bounded on region II which consists of the union of disks. Since F vanishes at the centers, there is a constant independent of n so that $|F| \le C|\zeta - n|$ on the n^{th} disk. In addition there is a $C' > 0$ independent of n so that $|\sin \pi\zeta| \ge C'|\zeta - n|$ on the n^{th} disk. Therefore $G \ = \ F/\sin \pi\zeta$ is bounded on region II.

Since the three regions exhaust the complex plane, it follows that G is uniformly bounded on $\mathbb{C}$. Liouville's Theorem implies that G is constant.

Take $\zeta = 2m + 1/2$ with $m \in \mathbb{Z}$ so $G(2m + 1/2) = F(2m + 1/2) \to 0$ as $m \to \infty$ by the Riemann-Lebesgue Lemma. Thus, the constant value of G must be 0. Therefore

$$F \ = \ G \sin \pi\zeta \ = \ 0 \sin \pi\zeta \ = \ 0\,.$$

Thus the Fourier transform of $\chi_{[-\pi,\pi]}\, f$ is identically equal to 0. Part **1.** implies that $\chi_{[-\pi,\pi]}\, f = 0$. Therefore $f = 0$ on $[-\pi, \pi]$. Since f is 2π-periodic it follows that $f = 0$. ∎

Proof of Theorem 3.1. Estimate (2.1) implies that the Fourier series on the right of (1.6) converges uniformly with all of its derivatives to an infinitely differentiable 2π periodic function $\tilde f$. Passing the integral through the sum shows that the Fourier coefficients of $\tilde f$ are equal to the Fourier coefficients of f. Thus $h := f - \tilde f$ is an infinitely differentiable smooth periodic function all of whose Fourier coefficients vanish. Part **2.** of Theorem 4.1 implies that $h = 0$. ∎

Remarks on the Theorem 3.1.

1. Part **1** implies that $\{e^{in\theta}/\sqrt{2\pi}\}$ is a complete orthonormal family in the square integrable periodic functions.

2. Convergence of the Fourier expansion of periodic distributions follows from the Theorem by a duality argument.

3. The division by sin in the proof is a close cousin of the use of the residue theorem applied to $g(z)/\sin \pi z$ to sum the series $\sum (-1)^n g(n)$. It will be employed again to prove the Sampling Theorem.

Exercise 5.1. *Derive the Fourier inversion formula from the preceding exercise of this section by showing that the only linear map $\mathcal{S}$ to itself that commute with x and d/dx are multiples of the identity.* **Hint.** Denote by L such a map. Use commutation with x show that $(Lu)(0)$ vanishes if $u(0)$ vanishes. Conclude that there is a constant c so that $(Lu)(0) = c\, u(0)$. Then, $Lu = c(x)\, u(x)$ with $c \in C^\infty$. Commuting with d/dx prove that $c' = 0$.

§6. Fourier integral representation from Fourier series. *

The Fourier integral representation follows from the Fourier series representation of periodic functions. We first present the idea of the derivation, then fill in the details.

Choose a function $\chi \in C^\infty(\mathbb{R})$ so that $0 \le \chi \le 1$, and,

$$\chi(x) \;=\; \begin{cases} 1 & \text{for} \quad -1 \le x \le 1\,, \\ 0 & \text{for} \quad\quad |x| \ge \pi\,. \end{cases}$$

For $L >> 1$, the truncated function $\chi(x/L)\; g(x)$ vanishes outside the the interval $[-\pi L, \pi L]$. Define $g_L(x)$ to be the $2\pi L$ periodic function which is equal to $\chi(x/L)\, g(x)$ on $[-\pi L, \pi L]$. The Fourier representation of g_L is then

$$g_L(x) \;=\; \sum a_n^L\, e^{inx/L}\,, \qquad a_n^L \;=\; \frac{1}{2\pi L} \int_{-\pi L}^{\pi L} e^{-inx/L}\; g_L(x)\; dx\,.$$

For large L one has

$$a_n^L \;=\; \frac{1}{2\pi L} \int_{-\infty}^{\infty} e^{-inx/L}\; \chi(x/L)\; g(x)\; dx \;\approx\; \frac{1}{L}\; \hat g(n/L)\,, \tag{6.1}$$

so the Fourier series representation of g_L yields

$$g_L(x) \;\approx\; \sum_n \hat g(n/L)\; e^{inx/L}\; \frac{1}{L}\,. \tag{6.2}$$

* This section is not needed for the Sampling Theorem.

For $|x| \le L$, $g = g_L$, so

$$g(x) \;\approx\; \sum_n \hat{g}(n/L)\; e^{inx/L}\; \frac{1}{L}\,, \qquad |x| \le L\,. \tag{6.3}$$

The right hand side of (6.3) is a Riemann sum with nodes at the points $\xi_n = n/L$ and $\Delta\xi = 1/L$. Therefore

$$g(x) \;\approx\; \int_{-\infty}^{\infty} \hat{g}(\xi)\; e^{in\xi}\; d\xi\,. \tag{6.4}$$

We will prove that the approximations become more and more accurate in the limit $L \to \infty$, thereby proving the Fourier integral representation (3.1), (3.2) from Fourier series.

Theorem 5.1. *If $g \in \mathcal{S}$ then $\hat{g} \in \mathcal{S}$ and the Fourier Inversion Formula (3.2) holds.*

Proof. That $\hat{g} \in \mathcal{S}$ has already been proved. We will justify the derivation of (3.2) from Fourier series by showing that the errors in the expressions indicated by $\approx$ tend to zero in the limit $L \to \infty$. There is an error committed in (6.1) that is equal to

$$\sum_n \left(\frac{1}{2\pi L}\; e^{inx/L} \int_{-\infty}^{\infty} e^{-inx/L}\; \big(\chi(x/L) - 1\big) g(x)\; dx \right). \tag{6.5}$$

The other error is the replacement of a Riemann sum by an integral passing from (6.3) to (6.4). This second error is equal to.

$$\int_{-\infty}^{\infty} \hat{g}(\xi)\; e^{ix\xi}\; d\xi \;-\; \sum_n \hat{g}(n/L)\; e^{inx/L}\; \frac{1}{L}\,. \tag{6.6}$$

Integrating by parts r times yields the estimate for the Fourier coefficient in (6.5),

$$\Big| \int_{-\infty}^{\infty} e^{-inx/L}\; \big(\chi(x/L) - 1\big) g(x)\; dx \Big| \;\le\; \frac{L^r}{|n|^r} \int_{-\infty}^{\infty} \Big| \frac{d^r}{dx^r} \Big(\big(\chi(x/L) - 1\big) g(x) \Big) \Big|\; dx\,.$$

Since $g \in \mathcal{S}$, the integral on the right hand side is $\le C(N)\, L^{-N}$ for any N. Choosing $r = 2$ one sees that the error (4.4) is $\le C(N)\, L^{-N+1}$, and in particular tends to zero.

For the error (6.6), let

$$\gamma(\xi) := e^{ix\xi}\, g(\xi)\,, \qquad \Delta\xi := \frac{1}{L}\,, \qquad \xi_n := n\Delta\xi\,, \qquad \text{and} \qquad I_n := [\xi_n, \xi_{n+1}]\,,$$

so the error is equal to

$$\int_{-\infty}^{\infty} \gamma(\xi)\; d\xi \;-\; \sum_n \gamma(\xi_n)\Delta\xi \;=\; \sum_n \Big(\int_{\xi_n}^{\xi_{n+1}} \gamma(\xi)\; d\xi \;-\; \gamma(\xi_n)\Delta\xi \Big)\,.$$

For x fixed, $\gamma \in \mathcal{S}$ so one has the estimate

$$\Big| \int_{\xi_n}^{\xi_{n+1}} \gamma(\xi)\; d\xi \;-\; \gamma(\xi_n)\Delta\xi \Big| \;\le\; OSC_{I_n}(\gamma)\, \Delta\xi \;\le\; \max_{I_n} |\gamma'|\; (\Delta\xi)^2 \;\le\; \frac{C(N)}{\langle \xi_n \rangle^N}\; (\Delta\xi)^2\,.$$

Taking $N=2$ and summing yields the estimate

$$\Big|\int_{-\infty}^{\infty} \gamma(\xi)\ d\xi\ -\ \sum_n \gamma(\xi_n)\Delta\xi\Big|\ \le\ C\,(\Delta\xi)^2\ \sum_n \frac{1}{1+(n/L)^2}\,.$$

Use

$$\sum_{|n|\ge 1} \frac{1}{1+(n/L)^2}\ \le\ 2\int_0^\infty \frac{1}{1+(x/L)^2}\ dx\ =\ 2\,L\int_0^\infty \frac{1}{1+y^2}\ dy\,,$$

to conclude that

$$\Big|\int_{-\infty}^{\infty} \gamma(\xi)\ d\xi\ -\ \sum_n \gamma(\xi_n)\Delta\xi\Big|\ \le\ \frac{C(x)}{L}\ \to\ 0\,,$$

as $L\to\infty$. ∎

Exercise 6.1. *Give details of the argument showing that G has removable singularities at $\zeta = n\pi$ with $n\in\mathbb{Z}$.*

Exercise 6.2. *The Weierstrass Approximation Theorem asserts that if f is a continuous function on an interval, then on that interval f can be uniformly approximated by polynomials. Prove this as follows. Show that it suffices to consider the interval $I=[-1,1]$. Given a continuous function on $[-1,1]$, show that there is a continuous 2π periodic extension to all of $\mathbb{R}$. Show that the periodic extension is the uniform limit of infinitely smooth 2π periodic functions.* **Hint.** Convolution with an approximate delta. Then approximate the smooth periodic function with a trigonometric polynomial by truncating the Fourier representation. Then approximate by a polynomial by approximating each exponential by a Taylor polynomial.

§7. The Sampling Theorem.

Definition. *A signal is called* **band limited** *if $\hat f$ is absolutely integrable and there is an $\Omega>0$ so that $\hat f(\omega)=0$ for all $|\omega|>\Omega$. Ω is called the* **band width**. *Such a signal is then given by*

$$f(t)\ =\ \int_{-\Omega}^{\Omega} \hat f(\omega)\ e^{i\omega t}\ d\omega\,. \tag{7.1}$$

The formula on the right hand side of (7.1) is defined for all complex numbers t. As in the proof of part **2** of Theorem 4.1, this defines an extension of f to an entire analytic function $f(\zeta)$.

The derivation of (5.3) shows that $f(\zeta)$ is of exponential growth in the sense that for all $t\in\mathbb{C}$

$$|f(\zeta)|\ \le\ e^{\Omega|\mathrm{Im}\,\zeta|}\int_{-\Omega}^{\Omega} |\hat f(\omega)|\ d\omega\,. \tag{7.2}$$

The analyticity of band limited signals has striking consequences. One is the unintuitive result that knowlege of the signal $f(t)$ on any arbitrarily short interval of time $a<t<b$ on the real axis determines its values at all times. This is a consequence of the unique continuation principle for analytic functions.

Discussion. It is standard engineering wisdom that in practice one cannot generate waves of arbitrarily short wavelength. Therefore all signals generated in the laboratory are band limited. It is also standard wisdom that no signals extend infinitely far into the past. There is a $T>0$ so that $f(t) = 0$ whenever t is real and $t < -T$. These two together imply that all signals are entire analytic functions that vanish on $]-\infty, -T] \subset \mathbb{R}$. The unique continuation principal for analytic functions implies that all such signals must vanish identically. Thus the only signal satisfying the two conditions of engineering wisdom is the identically vanishing signal! The resolution of this paradox is that while it is true that in the laboratory one can only control the spectrum on a bounded band, frequencies outside $-\Omega < \omega < \Omega$, are created. And, the insistence that there is absolutely no signal for large $|t|$ is also falacious. What is true is that any real signal can be well approximated by one of compact support in x. It can also be approximated by band limited signals. It cannot be approximated by a signal with both properties.

The proof of part **2** of Theorem 4.1 showed that a band limited signal with $\Omega = \pi$ with the property that the signal vanished at the points $x = n \in \mathbb{Z}$ must vanish identically. In this section we show that if the signal is sampled on a lattice with spacing smaller than 1, then the signal can be recovered from the sampled values by a stable summation formula.

More precisely, a band limited signal $f(t)$ can be recovered from its regularly spaced values

$$f(nL)\,, \qquad n \in \mathbb{Z}$$

providing that the spacing L of the sampling is sufficiently small. The hypothesis of the theorem is motivated by the example of the signal

$$f(t) = \sin \Lambda t = \frac{e^{i\Omega t} - e^{-i\Omega t}}{2\pi}$$

which is a limit of band limited signals with spectrum concentrated near the two points $\pm\Omega$. The signal vanishes at the points $n\pi/\Omega$ with spacing π/Ω.

This result dates at least to 1915 when Whittaker named it the *cardinal series*. Others rediscovered it including Shannon (1916-2001) in 1949. At that time the result entered into common use in information technology.

Definition. *The* **sinc** *function is defined by*

$$\mathrm{sinc}(t) \ := \ \frac{\sin \pi t}{\pi t}\,.$$

For $x = 0$ the value of sinc *is defined to be 1.*

Sampling Theorem 7.1. *If f is a band limited signal with band width Ω and $L < \pi/\Omega$ then f is reconstructed from its values sampled at the times $\{nL : n \in \mathbb{Z}\}$ as the sum of the convergent series,*

$$f(t) \ = \ \sum_{n=-\infty}^{\infty} \mathrm{sinc}\Big(\frac{t - nL}{L}\Big)\ f(nL)\,.$$

Remarks. 1. Since the sinc function vanishes at the integers, the function $\mathrm{sinc}\big((t-nL)/L\big)$ vanishes at mL for $m \neq n$. Therefore, For $t = mL$, the summands on the right with $n \neq m$ all vanish and the sampling identity is satisfied.

2. The convergence of the series is not obvious. However, if f has a bit of decay, for example if $(1 + |t|)^\beta f \in L^1(\mathbb{R})$ for some $\beta > 0$, it follows that $f(nL) = O(n^{-\alpha})$. * Since sinc decays

* Choose $\chi \in \mathcal{S}(\mathbb{R})$ with $\chi = 1$ on $[-L, L]$ and $\hat\chi \in C_0^\infty(\mathbb{R})$. Then $\chi \hat f = \hat f$. Therefore $f = \mathcal{F}^{-1}(\chi \hat f) = c\,\mathcal{F}^{-1}(\chi) * f$. The pointwise decay follows from the L^1 decay.

as $(1+|t|)^{-1}$ the terms of the series are then $O(n^{-1-\alpha})$, and, the series coverges absolutely and uniformly.

Proof. Replacing f by $\widetilde{f}(t) := f(Lt)$ changes the band width to $\Omega\, L$ and the sampling points to $n\mathbb{Z}$. So, it suffices to prove the result for $L = 1$ in which case the the limit on the bandwidth is $\Omega < \pi$.

The elegant convolution form of the Sampling Theorem for $L = 1$ is equivalent to

$$f(t) \;=\; \sum_{n=-\infty}^{n=\infty} \frac{(-1)^{n+1}\,\sin(\pi t)}{\pi\,(n-t)}\; f(n)\,,$$

which is the formula we prove.

For $t \notin \mathbb{Z}$ consider the function

$$g(z) = \frac{f(z)}{(z-t)\sin(\pi z)}\,.$$

g is analytic at all points of $\mathbb{C}$ except t and the roots, n, of $\sin(\pi z)$.

At those roots one has

$$\frac{d\sin(\pi t)}{dt}\Big|_{t=n} \;=\; \pi\cos(\pi n) \;=\; (-1)^n\pi \neq 0.\,,$$

Therefore the roots are simple so g has at worst a simple pole, and

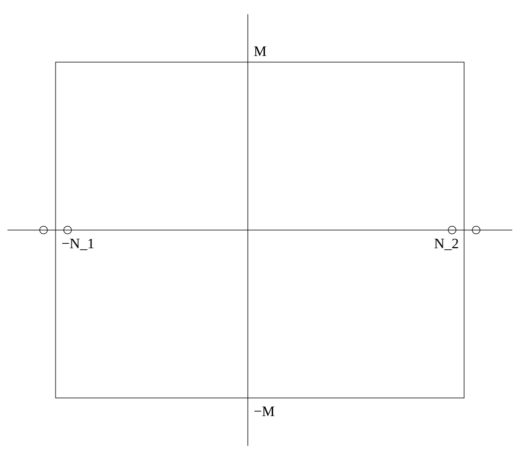
$$\mathrm{Res}(g,n) \;=\; \frac{(-1)^n\, f(n)}{\pi\,(n-t)}\,. \tag{7.3}$$

The function $g(z)$ also has a simple pole at t with

$$\mathrm{Res}(g,t) \;=\; \frac{f(t)}{\sin(\pi t)}\,. \tag{7.4}$$

For positive integers N_1, N_2 and M, define the rectangle

$$R_{N_1,N_2,M} \;:=\; \Big\{ z\in\mathbb{C} \;:\; -N_1-\frac{1}{2} < \mathrm{Re}\, z < N_2+\frac{1}{2} \quad\text{and}\quad |\mathrm{Im}\, z| < M \Big\}\,.$$



The vertical side on the right passes half way between the sampling points N_2 and N_2+1. The side on the left passes half way between $-N_1$ and $-N_1+1$.

Fix t. Consider $N_1, N_2, M > |t|+1$ The boundary does not hit any of the singularities of g so the Residue Theorem implies that for $t \notin \mathbb{Z}$

$$\frac{1}{2\pi i}\oint_{\partial R_{N_1,N_2,M}} g(\tau)\ d\tau = \mathrm{Res}(g,t) + \sum_{n=-N_1}^{n=N_2} \mathrm{Res}\,(g,n) = \frac{f(t)}{\sin(\pi t)} + \sum_{n=-N_1}^{n=N_2} \frac{(-1)^n\, f(n)}{\pi\,(n-t)}\,.$$

On $\partial R_{N_1,N_2,M}$ one has with constants indpendent of N_1, N_2, M,

$$\Big|\frac{1}{\sin(\pi z)}\Big| \le Ce^{-\pi|\mathrm{Im}\, z|}, \qquad |f(z)| \le Ce^{|\mathrm{Im}\, z|\Omega}, \qquad \Big|\frac{1}{z-t}\Big| \le \frac{C}{\min\{N_1,N_2,M\}}\,. \tag{7.5}$$

Let $M\to\infty$. The horizontal sides of $R_{N_1,N_2,M}$ have finite length and the integrand tends uniformly to zero since the decay of $1/\sin(\pi z/L)$ beats the growth of f because of the hypothesis $\pi > \Omega$. The integrands on the vertical sides decay exponentially for the same reason. Passing to the limit yields,

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\,\tau=N_2+1/2} g(\tau)\, d\tau - \frac{1}{2\pi i}\int_{\mathrm{Re}\,\tau=-N_1-1/2} g(\tau)\, d\tau \;=\; \frac{f(t)}{\sin(\pi t)} \;+\; \sum_{n=-N_1}^{n=N_2} \frac{(-1)^n\, f(n)}{\pi\,(n-t)}\,.$$

To complete the proof of the Theorem it suffices to show that each of the integrals on the left tend to zero as $N_1, N_2 \to \infty$. From (7.5), the absolute value of the integrand is bounded above by

$$\frac{C}{\min\{N_1,N_2\}}\ e^{-\pi|\mathrm{Im}\,\tau|(\pi-\Omega)}\,.$$

The sampling criterion $\pi > \Omega$ implies that the integrals are $O(1/\min\{N_1,N_2\})$. This completes the proof. ∎

Remark. The function $1/\sin\pi z$ with simple poles at the integers serves in a technique to evaluate sums by contour integration.

Exercise 7.1. *Apply the Cauchy integral formula to the function* $1/(z^2\sin\pi z)$ *in the disk of radius* $N+1/2$ *with* $\mathbb{N}\ni N\to\infty$ *to evaluate* $\sum_1^\infty(-1)^n n^{-2}$.

Exercise 7.2. *Apply the Cauchy integral formula to the function* $\cos\pi z/(z^2\sin\pi z)$ *to evaluate* $\sum_1^\infty n^{-2}$.