Fourier Series, Integrals, and Elementary Complex Analysis

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Outline. The Fourier series representation of analytic functions is derived from Laurent expansions. Elementary complex analysis is used to derive additional fundamental results in harmonic analysis including the representation of C^∞ periodic functions by Fourier series, the representation of rapidly decreasing functions by Fourier integrals, and Shannon's sampling theorem. In a sense, Cauchy's integral theorem is used instead of the usual approximate delta functions and Weierstrass' approximation theorem. Complex analysis yields four funcamental reslts;

1. Fourier series from Laurent expansions, Theorem 1.2,
2. Uniqueness of Fourier transforms, Theorem 4.1.1,
3. Uniqueness of Fourier series, Theorem 4.1.2,
4. Shannon's Theorem 5.1.

§1. Laurent series yield Fourier series.

A difficult thing to understand and/or motivate is the fact that arbitrary periodic functions have Fourier series representations. In this section we show that for periodic analytic functions, the representation follows from Laurent expansions. This suggests that any 2π-periodic function is a linear combination of the complex exponentials, e^{inx}.

A function $f(z)$ is periodic with period 2π when it satisfies

$$f(z+2\pi) = f(z) \tag{1.1}$$

for all z in its domain of definition. For this to make sense one requires that for any such z the points $z + 2\pi n$ also belong to the domain of definition when n is a positive or negative integer.

We are interested in functions which are defined at least for all real numbers $z = x + 0i$ and are analytic with domain of definition an open subset of $\mathbb{C}$.

It follows that f is defined on a neighborhood of the real axis which is invariant under translations by multiples of 2π and this implies that it contains a full strip

$$\{z \ : \ |\mathrm{Im}\, z| < a\}\,, \qquad a > 0\,. \tag{1.2}$$

To prove this, choose for each $\underline{x} \in [-\pi, \pi]$ an open box

$$B(\underline{x}, r) \ := \ \big\{z = x + iy \ : \ |x - \underline{x}| < r, \ \text{and}, \ |y| < r\big\}$$

contained in the domain of f. By compactness of $[-\pi,\pi]$, the interval is covered by a finite subfamily

$$[-\pi, \pi] \ \subset \ \cup_{j=1}^N \ B(x_j, r_j)\,.$$

Setting $a = \min\{r_j\}$ yields the desired conclusion.

Examples of periodic analytic functions. *The elementary functions* $\sin nz$, $\cos nz$, *and* $e^{\pm inz}$ *are the building blocks. Any finite linear combination is an example. . Nonlinear functions too, for example*

$$\frac{1}{1+\sin^2 z}$$

is analytic in any strip on which $\sin z \neq \pm i$*. An entire function* $h = \sum_0^\infty a_n\, z^n$ *yields the entire example*

$$h(e^{iz}) \;=\; \sum_0^\infty a_n\, e^{inz}\,.$$

The last example can be modified to yield the general case as follows. Consider the mapping

$$w \;=\; e^{iz}\,. \tag{1.3}$$

It maps the strip (2) in the complex z plane to the annulus

$$\{w \;:\; e^{-a} < |w| < e^{a}\} \tag{1.4}$$

in the w plane. It maps the real axis in the z plane infinitely often around the unit circle in the w plane. The preimages of a point $w = e^{i\theta}$ are the points $z = \theta + 2\pi n$ with $n \in \mathbb{Z}$. Since the derivative dw/dz is nowhere zero it follows that the mapping is locally invertible with analytic inverse. The local inverses are branches of the function $z = (\ln w)/i$.

Theorem 1.1. *The correspondence*

$$f(z) \;=\; g(e^{iz}) \tag{1.5}$$

establishes a one to one correspondence between the 2π *periodic analytic functions* $f(z)$ *in the strip (1.2) and the analytic functions* $g(w)$ *on the annulus (1.4).*

Proof. That each such g yields an analytic periodic f on the strip and that distinct functions g yield distinct f is clear. Next show that every periodic analytic function on the strip has such a representation. Suppose that f is analytic and periodic in the strip (1.2). For each point w in the annulus (1.4) the preimages z under the map (1.3) lie in the strip and differ by integer multiples of 2π. Thus, the function f has the same value at all the preimages. It follows that a function g on the annulus is well defined by the formula $g(w) = f(z)$ since it does not matter which z one takes.

For any $\underline{w}$ choose a preimage $\underline{z}$. The Inverse Function Theorem implies that w has a local inverse $z = F(w)$ analytic on a neighborhood of $\underline{w}$ and satisfying $F(\underline{w}) = \underline{z}$. Identity (1.5) on a neighborhood of $\underline{z}$ implies that on a neighborhood of $\underline{w}$ one must have. $g(w) = f(F(w))$. This proves that g is unquely determined on a neighborhood of $\underline{w}$ and therefore on the annulus by the unique continuation principal. In addition the formula $g = f \circ F$ proves that g is analytic on a neighborhood of $\underline{w}$. Since $\underline{w}$ is arbitrary, is therefore g is analytic so provides the desired representation of f.

∎

Theorem 1.2. *If* $f(z)$ *is a* 2π *periodic analytic function in the strip (1.2) then* f *has a Fourier series representation*

$$f(z) \;=\; \sum_{n=-\infty}^{\infty} c_n\, e^{inz}\,, \tag{1.6}$$

with coefficients given by the formulas

$$c_n \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, e^{-in\theta}\, d\theta\,. \tag{1.7}$$

Proof. Choose g so that (1.5) holds. Then use the Laurent expansion of g

$$g(w) \;=\; \sum_{-\infty}^{\infty} c_n\, w^n\,, \qquad c_n \;=\; \frac{1}{2\pi i}\oint_{|w|=1} \frac{g(w)}{w^{n+1}}\; dw\,. \tag{1.8}$$

Since $f(z) = g(e^{iz})$, one has

$$f(z) \;=\; \sum_{-\infty}^{\infty} c_n\, (e^{iz})^n$$

which is formula (1.6).

Parameterizing the curve $|w| = 1$ by $w = e^{i\theta}$ with $0 \le \theta \le 2\pi$, one has $dw = i\, w\, d\theta$ and the formula for c_n becomes

$$c_n \;=\; \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(e^{i\theta})}{w^{n+1}}\; i\, w\, d\theta \;=\; \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)}{w^n}\; d\theta\,,$$

which proves (1.7). ∎

§2. Fourier series for nonanalytic periodic functions.

Using Laurent expansions, we have shown that every 2π periodic function which is analytic in a neighborhood of the real axis has a Fourier series representation (1.6)-(1.7). It is worth noting that if (1.6) holds then multiplying by e^{-imx} then integrating over $[-\pi, \pi]$ yields the formula (1.6) for the coefficients since

$$\int_{-\pi}^{\pi} e^{-imx}\; e^{inx}\; dx \;=\; \begin{cases} 2\pi & \text{when} \quad m = n \\ 0 & \text{when} \quad m \ne n\,. \end{cases}$$

Periodic functions which need not be analytic have Fourier expansion of the same form. This too can be proved by purely complex variable methods.

The smoother are the functions, the more rapidly decreasing are the coefficients c_n and the faster is the convergence in (1.6). For analytic f the c_n are exponentially decreasing as $|n| \to \infty$,

$$|c_n| \;\le\; C\, e^{-a|n|}\,, \qquad c > 0\,.$$

Exercises. 1. *Prove the exponential decay for $n > 0$ starting from the formula for c_n in (1.8). For $1 < b < e^a$ move the contour to $|z| = b$ using Cauchy's Theorem. On that contour, $1/w^{n+1}$ is exponentially small as $n \to \infty$. Perform an analogous estimate to treat $n < 0$.*

2. *Conversely, if f is periodic and given by (1.6) with c_n decaying exponentially in $|n|$ prove that f has an analytic continuation to a strip containing the real axis.* **Hint.** The Fourier series is uniformly convergent on thin strips.

For infinitely differentiable periodic f the c_n decrease faster than any negative power of $|n|$, that is for each N there is a constant $C(N)$ so that

$$\forall\, N,\ \exists\, C_N,\ \forall\, n, \quad |c_n| \;\le\; \frac{C_N}{(1 + |n|)^N}\,, \tag{2.1}$$

This is a consequence of the formula for the Fourier coefficients of the derivative,

$$c_n(f') \;=\; in\, c_n(f)\,, \tag{2.2}$$

valid for example if f is continuously differentiable. To prove (2.2), integrate by parts with boundary terms cancelling by periodicity to give,

$$2\pi\, c_n(f') \;=\; \int_{-\pi}^{\pi} f'(x)\; e^{-inx}\; dx \;=\; -\int_{-\pi}^{\pi} f(x)\; \frac{de^{-inx}}{dx}\; dx \;=\; 2\pi\, in\, c_n(f)\,. \tag{2.3}$$

Therefore if $f \in C^N$,

$$(-in)^N c_n(f) \;=\; c_n(f^{(N)}) \;\le\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|\frac{d^N f}{dx^N}(x)\Big|\; dx \tag{2.4}$$

which implies (2.1).

In these cases convergence of each differentiated series is uniform. For f which are merely square integrable, one has only $\sum |a_n|^2 \;<\; \infty$, the convergence is in the root mean square sense, and the differentiated series need not converge. For periodic distributions in the sense of Schwartz, the convergence is in the sense of distributions. The C^∞ result implies the others (see remarks below). In this note, we show how the C^∞ result follows from elementary complex analysis. We use neither approximate delta functions nor the Weierstrass approximation theorem. The latter is derived in Exercise 2 of §4. It is worth recalling the fact that a function which is not regular, for example a periodic square wave or sawtooth can be expressed as a sum of the regular functions $c_n e^{in\theta}$ was considered surprising.

Historically, many examples of expansions (1.6) were discovered before it was realized how general was the phenomenon. For example if $|a| < 1$, one has

$$\frac{1}{1 + a\sin\theta} \;=\; \sum_{n=0}^{\infty} \big(a\sin\theta\big)^n\,.$$

It was Fourier who uncovered the fact that the representations were general and their utility in analysing differential equations. This preceded the flowering of complex analysis. It would have been logically very appealing to have discovered the theory of Fourier series starting from Laurent expansions as we present it here.

Theorem 2.1. *If f is an infinitely differentiable 2π periodic function then the representation (1.6), (1.7) is valid. The Fourier coefficients (1.7) satisfy the rapid decay estimate (2.1) so the series and all differentiated series converge uniformly on $\mathbb{R}$.*

§3. The Fourier transform.

Our proof is intimately entangled with the Fourier transform representation

$$g(x) \;=\; \int_{-\infty}^{\infty} \hat g(\xi)\, e^{ix\xi}\; d\xi \;:=\; \mathcal{F}(g)(\xi)\,, \tag{3.1}$$

$$\hat g(\xi) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat g(x)\, e^{-ix\xi}\; dx\,, \tag{3.2}$$

for functions g defined on $\mathbb{R}$ and tending to zero at $\pm\infty$. Using contour integration, we have verified this reciprocal relation in the two concrete cases of

$$g(x) = e^{-a|x|}, \quad a > 0\,, \qquad \hat g(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+\xi^2}\,, \tag{3.3}$$

and

$$g(x) = e^{-x^2/2}\,, \qquad \hat g(\omega) = e^{-\xi^2/2}\,. \tag{3.4}$$

The standard proofs of (1.6), (1.7) proceed beginning with one of these two examples, and then constructs an approximate delta function for which the relations are true and then concludes by convolution. The pages that follow give an alternative approach in the spirit of the proof of Shannon's sampling theorem which is discussed in a final section.

A convenient class of functions for studying the Fourier transform is the Schwatrz class $\mathcal{S}$ consisting of those g so that for all $0 < n, m$ there is a $C = C(n, m)$ so that

$$\forall x \in \mathbb{R}, \qquad \Big|\frac{d^m g}{dx^m}\Big| \;\le\; C\,\langle x\rangle^{-n}\,.$$

For such g, an integration by parts as in (2.3) shows that

$$\mathcal{F}(g') \;=\; i\xi\ \hat g\,.$$

Differentiating the definition of $\hat g$ yields

$$\frac{d}{d\xi}\,\hat g = \mathcal{F}(-ix\, g)\,.$$

It follows that the Fourier transform of a function in $\mathcal{S}$ belongs to $\mathcal{S}$ so that in (3.1), (3.2) the integrals are very rapidly convergent.

The Fourier integral representation follows from the Fourier series representation of periodic functions. Choose a function $\chi \in C^\infty(\mathbb{R})$ so that $0 \le \chi \le 1$, and,

$$\chi(x) \;=\; \begin{cases} 1 & \text{for} \qquad -1 \le x \le 1\,, \\ 0 & \text{for} \qquad\quad |x| \ge \pi\,. \end{cases}$$

For $L >> 1$, the truncated function $\chi(x/L)\ g(x)$ vanishes outside the the interval $[-\pi L, \pi L]$. Define $g_L(x)$ to be the $2\pi L$ periodic function which is equal to $\chi(x/L)\,g(x)$ on $[-\pi L, \pi L]$. The Fourier representation of g_L is then

$$g_L(x) \;=\; \sum a_n^L\, e^{inx/L}\,, \qquad a_n^L \;=\; \frac{1}{2\pi L}\int_{-\pi L}^{\pi L} e^{-inx/L}\ g_L(x)\ dx\,.$$

For large L one has

$$a_n^L \;=\; \frac{1}{2\pi L}\int_{-\infty}^{\infty} e^{-inx/L}\ \chi(x/L)\ g(x)\ dx \;\approx\; \frac{1}{L}\ \hat g(n/L)\,, \tag{3.6}$$

so the Fourier series representation of g_L yields

$$g_L(x) \;\approx\; \sum_n \hat g(n/L)\ e^{inx/L}\ \frac{1}{L}\,. \tag{3.7}$$

For $|x| \le L$, $g = g_L$, so

$$g(x) \;\approx\; \sum_n \hat g(n/L)\ e^{inx/L}\ \frac{1}{L}\,, \qquad |x| \le L\,. \tag{3.8}$$

The right hand side of (3.8) is a Riemann sum with nodes at the points $\xi_n = n/L$ and $\Delta\xi = 1/L$. Therefore

$$g(x) \;\approx\; \int_{-\infty}^{\infty} \hat{g}(\xi)\; e^{in\xi}\; d\xi\,. \tag{3.9}$$

At the end of the next section we will prove that the approximations become more and more accurate in the limit $L \to \infty$, thereby proving the Fourier integral representation (3.4), (3.5) from Fourier series.

The analysis requires the following fundamental result.

Riemann-Lebesgue Lemma 3.1. *If* $\int_{-\infty}^{\infty} |g(x)|\,dx < \infty$, *then*

$$\lim_{|\xi|\to\infty} \hat{g}(\xi) \;=\; 0\,.$$

Proof. For $\epsilon > 0$, choose $\psi \in \mathcal{S}$ so that

$$\int_{-\infty}^{\infty} \big|g(x) - \psi(x)\big|\; dx \;<\; \epsilon\,.$$

Then

$$\forall \xi \in \mathbb{R}\,, \qquad \big|\hat{g}(\xi) \;-\hat{\psi}(\xi)\big| \;<\; \frac{\epsilon}{2\pi}\,.$$

Since $\hat{\psi}(\xi) \to 0$ as $|\xi| \to \infty$, one has

$$\limsup_{|\xi|\to\infty}\; |\hat{g}(\xi)| \;\leq\; \frac{\epsilon}{2\pi}\,.$$

Since this is true for any $\epsilon > 0$ the proof is complete. ∎

This result applied to $g = \chi_{[-\pi,\pi]}(x)\,f(x)$ with g a 2π-periodic function implies that the Fourier coefficients of an absolutely integrable periodic function tend to zero as $n \to \infty$.

§4. Uniqueness of Fourier transforms, proof of Theorem 2.1.

The key step in our proof of (1.6), (1.7) is to prove that if a C^∞ periodic function f has all its Fourier coefficients equal to zero, then the function vanishes. Similarly if an absolutely integrable function g on $\mathbb{R}$, has Fourier transform $\hat{g}$ identically equal to 0, then $g = 0$. Equivalently, if two periodic functions f_1 and f_2 have the same Fourier coefficients, then $f_1 = f_2$, and if g_1 and g_2 are absolutely integrable functions on $\mathbb{R}$ have the same Fourier transforms, then $g_1 = g_2$. These equivalences follow from applying the preceding assertions to $f = f_1 - f_2$ and $g_1 - g_2$ respectively.

Theorem 4.1. 1. *If $g(x)$ is an absolutely integrable function on $\mathbb{R}$ whose Fourier transform is identically equal to zero, then $g = 0$.*

2. *If $f(x)$ is a 2π periodic function absolutely integrable over each period whose Fourier coefficients defined by (1.6) are all equal to zero, then $f = 0$.*

Proof. 1. Write

$$\hat{g} = F_- + F_+\,, \qquad F_-(\zeta) := \int_{-\infty}^{0} g(x)\,e^{-ix\zeta}\,dx\,, \quad F_+(\zeta) := \int_{0}^{\infty} g(x)\,e^{-i\zeta x}\,dx\,.$$

The function $F_\pm$, is continuous and uniformly bounded on $\{\pm \mathrm{Im}\,\zeta \le 0\}$. Differentiating under the integral shows that $F_\pm$ is analytic in the interior, $\{\pm\mathrm{Im}\,\zeta < 0\}$. On the boundary of the two half spaces one has $F_+ + F_- = F = 0$. Therefore the function

$$H(\zeta) \;:=\; \pm F_\pm \quad \text{when} \quad \pm\,\mathrm{Im}\,\zeta \ge 0\,,$$

is holomorphic and uniformly bounded on each half space. The relation $F_+ = -F_-$ on the real axis establishes the continuity of H at those. It follows that H is a bounded entire function. Liouville's Theorem implies that H is constant.

The Riemann-Lebesgue Lemma implies that H tends to zero at infinity on the real axis, so the constant must be 0. Therefore $H = 0$. In particular,

$$0 = H(0) = F_+(0) = \int_{-\infty}^0 g(\theta)\, d\theta\,.$$

For any $a \in \mathbb{R}$, the function $g(x-a)$ also has vanishing Fourier Transform since

$$\int g(x-a)\; e^{-ix\xi}\; dx \;=\; \int f(x)\; e^{-i(x+a)\xi}\; dx \;=\; e^{-ia\xi} \int g(x)\; e^{-ix\xi}\; dx \;=\; 0\,.$$

Therefore

$$0 \;=\; \int_{-\infty}^0 f(x-a)\; dx \;=\; \int_{-\infty}^a f(x)\; dx\,.$$

Since this is true for all a it follows that $f = 0$.

2. For $\zeta \in \mathbb{C}$, introduce the Fourier Transform

$$F(\zeta) \;:=\; \int_{-\pi}^{\pi} g(\theta)\; e^{-i\theta\zeta}\; d\theta\,, \qquad \zeta = \xi + i\eta\,, \quad \xi, \eta \in \mathbb{R}\,. \tag{4.1}$$

The vanishing of the Fourier coefficients yields

$$F(n) \;=\; 0\,, \qquad \text{for all} \quad n \in \mathbb{Z}\,. \tag{4.2}$$

Differentiating under the integral sign shows that F has partial derivatives of all order with respect to ξ and η and

$$\frac{\partial F}{\partial \xi} \;=\; \int_{-\pi}^{\pi} g(\theta)\; \frac{\partial e^{-i\theta\zeta}}{\partial \xi}\; d\theta\,, \qquad \frac{\partial F}{\partial \eta} \;=\; \int_{-\pi}^{\pi} g(\theta)\; \frac{\partial e^{-i\theta\zeta}}{\partial \eta}\; d\theta\,.$$

Since $e^{-i\theta\zeta}$ is an analytic function of ζ it satisfies the Cauchy-Riemann equation

$$\frac{\partial e^{-i\theta\zeta}}{\partial \xi} \;=\; \frac{1}{i}\,\frac{\partial e^{-i\theta\zeta}}{\partial \eta}\,.$$

Therefore F also satisfies the Cauchy-Riemann equations, so F is an entire analytic function. Since $e^{-i\theta\zeta} = e^{-i\theta\xi}e^{\eta\theta}$, it follows that for $\theta \in [-\pi, \pi]$,

$$|e^{-i\theta\zeta}| \;\le\; e^{\pi|\mathrm{Im}\,\zeta|}\,, \qquad \text{so,} \qquad |F(\zeta)| \;\le\; \frac{e^{\pi|\mathrm{Im}\,\zeta|}}{2\pi} \int_{-\pi}^{\pi} |g(\theta)|\; d\theta\,. \qquad (4.3)\,.$$

Define

$$G(\zeta) \ := \ \frac{F(\zeta)}{\sin \pi\zeta}\,,$$

so G is analytic except possibly for isolated singularities at the zeroes $\zeta = n \in \mathbb{Z}$ of $\sin \pi\zeta$. These are simple zeroes of $\sin z$ and F vanishes at these points by (4.2). Consequently, G has a removable singularity at each of these points (Exercise 1.) Therefore, G is a 2π-periodic entire analytic function.

For $|\mathrm{Im}\,\zeta| \geq 1$, there is a $C > 0$, so that $|\sin \pi\zeta| \geq Ce^{\pi|\mathrm{Im}\,\zeta|}$. Therefore (4.3) shows that G is uniformly bounded on $\{|\mathrm{Im}\,\zeta| \geq 1\}$.

G is continuous so is uniformly bounded on the compact set $\{-\pi \leq \mathrm{Re}\,\zeta \leq \pi\,,\ |\mathrm{Im}\,\zeta| \leq 1\}$. Since G is 2π periodic, it follows that G is uniformly bounded on $\{|\mathrm{Im}\,\zeta| \leq 1\}$.

The preceding paragraphs show that G is uniformly bounded on $\mathbb{C}$, so Liouville's Theorem implies that G is constant.

Take $\zeta = 2m + 1/2$ with $m \in \mathbb{Z}$ so $G(2m + 1/2) = F(2m + 1/2) \to 0$ as $m \to \infty$ by the Riemann-Lebesgue Lemma. Thus, the constant value of G must be 0. Therefore $F = G \sin \pi\zeta = 0$.

Thus the Fourier transform of $\chi_{[-\pi,\pi]}\, g$ is identically equal to 0, so part **1** implies that $\chi_{[-\pi,\pi]}\, g = 0$. Therefore $g = 0$ on $[-\pi, \pi]$. Since g is 2π-periodic it follows that $g = 0$. ∎

Proof of Theorem 2.1. Estimate (2.1) implies that the Fourier series on the right of (1.6) converges uniformly with all of its derivatives to an infinitely differentiable 2π periodic function $\tilde{f}$. Passing the integral through the sum shows that the Fourier coefficients of $\tilde{f}$ are equal to the Fourier coefficients of f. Thus $h := f - \tilde{f}$ is an infinitely differentiable smooth periodic function all of whose Fourier coefficients vanish. Part **2.** of Theorem 3 implies that $h = 0$. ∎

Remarks on the Theorem 2.1.

1. Part **1** implies that $\{e^{in\theta}/\sqrt{2\pi}\}$ is a complete orthonormal family in the square integrable periodic functions.

2. Convergence of the Fourier expansion of periodic distributions follows from the Theorem by a duality argument.

3. The division by $\sin \pi\zeta$ recalls the method for evaluating infinite sums using the Residue Theorem.

4. The division by $\sin \pi\zeta$ will be used to prove Shannon's Sampling Theorem in the next section.

Now that the Fourier series representation has been proved we can prove the Fourier Inversion Formula.

Theorem 4.2. *If $g \in \mathcal{S}$ then $\hat{g} \in \mathcal{S}$ and the Fourier Inversion Formula (3.2) holds.*

Proof. That $\hat{g} \in \mathcal{S}$ has already been proved. We justify the derivation of (3.2) from Fourier series by showing that the errors in the expressions indicated by $\approx$ tend to zero in the limit $L \to \infty$. There is an error committed in (3.6) which is equal to

$$\sum_n \left(\frac{1}{2\pi L}\ e^{inx/L} \ \int_{-\infty}^{\infty} e^{-inx/L}\ \big(\chi(x/L) - 1\big) g(x)\ dx \right). \tag{4.4}$$

The other error is the repacement of a Riemann sum by an integral passing from (3.8) to (3.9). This second error is equal to.

$$\int_{-\infty}^{\infty} \hat{g}(\xi)\ e^{ix\xi}\ d\xi \ - \ \sum_n \hat{g}(n/L)\ e^{inx/L}\ \frac{1}{L}\,. \tag{4.5}$$

Integrating by parts r times yields the estimate for the Fourier coefficient in (4.4),

$$\Big|\int_{-\infty}^{\infty} e^{-inx/L}\ \big(\chi(x/L)-1\big)g(x)\ dx\Big| \ \le\ \frac{L^r}{|n|^r}\ \int_{-\infty}^{\infty}\Big|\frac{d^r}{dx^r}\Big(\big(\chi(x/L)-1\big)g(x)\Big)\Big|\ dx\,.$$

Since $g\in\mathcal{S}$, the integral on the right hand side is $\le C(N)\,L^{-N}$ for any N. Choosing $r=2$ one sees that the error (4.4) is $\le C(N)\,L^{-N+1}$, and in particular tends to zero.

For the error (4.5), let

$$\gamma(\xi):=e^{ix\xi}\,g(\xi)\,,\qquad \Delta\xi:=\frac{1}{L}\,,\qquad \xi_n:=n\Delta\xi\,,\qquad\text{and}\qquad I_n:=[\xi_n,\xi_{n+1}]\,,$$

so the error is equal to

$$\int_{-\infty}^{\infty}\gamma(\xi)\ d\xi\ -\ \sum_n \gamma(\xi_n)\Delta\xi\ =\ \sum_n\Big(\int_{\xi_n}^{\xi_{n+1}}\gamma(\xi)\ d\xi\ -\ \gamma(\xi_n)\Delta\xi\Big)\,.$$

For x fixed, $\gamma\in\mathcal{S}$ so one has the estimate

$$\Big|\int_{\xi_n}^{\xi_{n+1}}\gamma(\xi)\ d\xi\ -\ \gamma(\xi_n)\Delta\xi\Big|\ \le\ OSC_{I_n}(\gamma)\,\Delta\xi\ \le\ \max_{I_n}|\gamma'|\ (\Delta\xi)^2\ \le\ \frac{C(x,N)}{\langle\xi_n\rangle^N}\ (\Delta\xi)^2\,.$$

Taking $N=2$ and summing yields the estimate

$$\Big|\int_{-\infty}^{\infty}\gamma(\xi)\ d\xi\ -\ \sum_n\gamma(\xi_n)\Delta\xi\Big|\ \le\ C\,(\Delta\xi)^2\ \sum_n\frac{1}{1+(n/L)^2}\,.$$

Use

$$\sum_{|n|\ge1}\frac{1}{1+(n/L)^2}\ \le\ 2\int_0^{\infty}\frac{1}{1+(x/L)^2}\ dx\ =\ 2\,L\int_0^{\infty}\frac{1}{1+y^2}\ dy\,,$$

to conclude that

$$\Big|\int_{-\infty}^{\infty}\gamma(\xi)\ d\xi\ -\ \sum_n\gamma(\xi_n)\Delta\xi\Big|\ \le\ \frac{C(x)}{L}\ \to\ 0\,,$$

as $L\to\infty$. ▮

Exercises. 1. *Give details of the argument showing that G has removable singularities at $\zeta=n\pi$ with $n\in\mathbb{Z}$.*

2. *The Weierstrass Approximation Theorem asserts that if f is a continuous function on an interval, then on that interval f can be uniformly approximated by polynomials. Prove this as follows. Show that it suffices to consider the interval $I=[-1,1]$. Given a continuous function on $[-1,1]$, show that there is a continuous 2π periodic extension to all of $\mathbb{R}$. Show that the periodic extension is the uniform limit of infinitely smooth 2π periodic functions.* **Hint.** Convolution with an approximate delta. Then approximate the smooth periodic function with a trigonometric polynomial by truncating the Fourier representation. Then approximate by a polynomial by approximating each exponential by a Taylor polynomial.

§5. Band limited signals

A signal is called **band limited** if there is an $\Omega > 0$ so that $\hat{f}(\omega) = 0$ for all $|\omega| > \Omega$. Such a signal is then given by

$$f(t) \;=\; \int_{-\Omega}^{\Omega} \hat{f}(\omega)\; e^{i\omega t}\; d\omega\,, \tag{5.1}$$

We suppose that $\hat{f}(\omega)$ in an absolutely integrable function. Ω is called the band width.

The formula on the right hand side of (5.1) is defined for all complex numbers t. This defines an extension of f to a function on the whole complex plane $\mathbb{C}$. As in the proof of part **2** of Theorem 3.1, every band limited signal f is entire analytic function of $t \in \mathbb{C}$. If its spectrum is contained in $[-\Omega, \Omega]$ as in (5.1) and $\hat{f}$ is an integrable function then f is of exponential growth in the sense that for all $t \in \mathbb{C}$

$$|f(t)| \;\le\; e^{\Omega |\mathrm{Im}\, t|} \int_{-\Omega}^{\Omega} |\hat{f}(\omega)|\; \frac{d\omega}{\sqrt{2\pi}}\,. \tag{5.2}$$

The analyticity of band limited signals has striking consequences. One is the unintuitive result that knowlege of the signal $f(t)$ on any arbitrarily short interval of time $a < t < b$ on the real axis determines its values at all times. This is a straightforward consequence of the unique continuation principal for analytic functions. If there were a second signal $\tilde{f}$ which agreed with f for $a < t < b$ then the difference $f - \tilde{f}$ would be entire and would vanish on $a < t < b$ and therefore would vanish identically.

Discussion. It is standard engineering wisdom that in practice one cannot generate waves of arbitrarily short wavelength. Therefore all signals generated in the laboratory are band limited. It is also standard wisdom that no signals extend infinitely far into the past. That is there is a $T > 0$ so that $f(t) = 0$ whenever t is real and $t < -T$. These two together imply that all signals are entire analytic functions which vanish on $]-\infty, -T] \subset \mathbb{R}$. The unique continuation principal for analytic functions implies that all such signals must vanish identically. Thus the only signal satisfying the two conditions of engineering wisdom is the identically vanishing signal! The resolution of this paradoxical result is that neither the limit to the band $-\Omega < \omega < \Omega$ nor to the time interval $t < -T$ is exact.

The proof of part **2** of Theorem 3.1 showed that a band limited signal with $\Omega = \pi$ with the property that the signal vanished at the points $x = n \in \mathbb{Z}$ must vanish identically. In this section we show that the signal can be recovered by a stable formula from its values on lattices with spacing smaller than 1.

More generally, a band limited signal $f(t)$ can be recovered from its regularly spaced values

$$f(nL)\,, \qquad n \in \mathbb{Z}$$

providing that the spacing L of the sampling is sufficiently small. The hypothesis of the theorem is motivated by the example of the signal

$$f(t) = \sin \Lambda t = \frac{e^{i\Omega t} - e^{-i\Omega t}}{2\pi}$$

which is a limit of band limited signals with spectrum concentrated near the two points $\pm\Omega$. The limiting signal vanishes at the points $n\pi/\Omega$ with spacing π/Ω.

Shannon's Sampling Theorem 5.1. *If f is a band limited signal with band width Ω and $L < \pi/\Omega$ then f is reconstructed from its values sampled at the times $\{nL \,:\, xn \in \mathbb{Z}\}$ as the sum of the convergent series*

$$f(t) \;=\; \sum_{n=-\infty}^{n=\infty} \frac{(-1)^{n+1} L\, \sin(\pi t/L)}{\pi\,(nL - t)}\; f(nL)\,.$$

Remark. For $t = mL$, the summands on the right with $n \neq m$ all vanish. The summand with $n = m$ is defined by setting

$$\frac{L\,\sin(\pi t/L)}{\pi\,(nL-t)}\Big|_{t=nL} \;=\; -1\,.$$

With that natural definition, the sampling identity is trivially satisfied for $t \in L\mathbb{Z}$.

Proof. For $t \notin L\mathbb{Z}$ consider the function

$$g(z) = \frac{f(z)}{(z-t)\sin(\pi z/L)}\,.$$

The Paley-Wiener Theorem shows that f is entire and it follows that g is analytic at all points of $\mathbb{C}$ except t and the roots, nL, of $\sin(\pi z/L)$.

At those roots one has

$$\frac{d\sin(\pi t/L)}{dt}\Big|_{t=nL} \;=\; \frac{\pi}{L}\cos(\pi n) \;=\; \frac{(-1)^n\pi}{L} \neq 0.\,,$$

Therefore the roots are simple so g has at worst a simple pole, and

$$\operatorname{Res}(g, nL) \;=\; \frac{(-1)^n\,L\,f(nL)}{\pi\,(nL-t)}\,. \tag{5.3}$$

The function $g(z)$ also a simple pole at t with

$$\operatorname{Res}(g, t) \;=\; \frac{f(t)}{\sin(\pi t/L)}\,. \tag{5.4}$$

For positive integers N_1, N_2 and M, define a rectangle

$$R_{N_1,N_2,M} \;:=\; \Big\{ z \in \mathbb{C} \;:\; -N_1 - \frac{L}{2} < \operatorname{Re} z < N_2 + \frac{L}{2} \quad \text{and} \quad |\operatorname{Im} z| < M \Big\}\,.$$

When N_1, N_2 and M are larger than $|t|$, the boundary does not hit any of the singularities of g so the Residue Theorem implies that for $t \notin L\mathbb{Z}$

$$\frac{1}{2\pi i}\oint_{\partial R_{N_1,N_2,M}} g(\tau)\,d\tau \;=\; \operatorname{Res}(g,t) \;+\; \sum_{n=-N}^{n=N} \operatorname{Res}(g, nL) \;=\; \frac{f(t)}{\sin(\pi t/L)} \;+\; \sum_{n=-N_1}^{n=N_2} \frac{(-1)^n L\,f(nL)}{\pi\,(nL-t)}\,.$$

On $\partial R_{N_1,N_2,M}$ one has with constants indpendent of N_1, N_2, M but depending on t,

$$\Big|\frac{1}{\sin(\pi z/L)}\Big| \le C e^{-\pi|\operatorname{Im} z|/L}, \qquad |f(z)| \le C e^{|\operatorname{Im} z|\Omega}, \qquad \Big|\frac{1}{z-t}\Big| \le \frac{C}{\min\{N_1,N_2,M\}}\,. \tag{5.5}$$

Fix t and $N_1, N_2 > |t|$ and let $M \to \infty$. The horizontal sides of $R_{N_1,N_2,M}$ have finite length and the integrand tends uniformly to zero since the decay of $1/\sin(\pi z/L)$ beats the growth of f because of the hypothesis $\pi/L > \Omega$. Thus

$$\frac{1}{2\pi i}\int_{\operatorname{Re}\tau = N_2 + L/2} g(\tau)\,d\tau - \frac{1}{2\pi i}\int_{\operatorname{Re}\tau = -N_1 - L/2} g(\tau)\,d\tau \;=\; \frac{f(t)}{\sin(\pi t/L)} \;+\; \sum_{n=-N_1}^{n=N_2} \frac{(-1)^n L\,f(nL)}{\pi\,(nL-t)}\,.$$

To complete the proof of the Theorem it suffices to show that each of the integrals on the left tend to zero as $N_1, N_2 \to \infty$. From (5.5), the absolute value of the integrand is bounded above by

$$\frac{C}{\min\{N_1,N_2\}}\; e^{-\pi|\operatorname{Im}\tau|(\frac{\pi}{L}-\Omega)}\,.$$

Since $\frac{\pi}{L} - \Omega > 0$, the integrals are $O(1/\min\{N_1,N_2\})$ completing the proof. ∎