Laurent Series Yield Fourier Series

A difficult thing to understand and/or motivate is the fact that arbitrary periodic functions have Fourier series representations. In this short note we show that for periodic functions which are analytic the representation follows from basic facts about Laurent series.

§1. Fourier series of analytic periodic functions.

A function $f(z)$ is periodic with period 2π if it satisfies

$$f(z+2\pi) = f(z) \tag{1}$$

for all z in its domain of definition. For this to make sense one requires that for any such z the points $z + 2\pi n$ also belong to the domain of definition when n is a positive or negative integer.

We are interested in functions which are defined at least for all real numbers $z = x + 0i$ and are analytic with domain of definition an open subset of $\mathbf{C}$. It follows that f is defined on a neighborhood of the real axis which is invariant under translations by multiples of 2π and this implies (exercise) that it contains a full strip

$$\{z \ : \ |\mathrm{Im}\, z| < a\}\,, \qquad a > 0\,. \tag{2}$$

Examples of periodic analytic functions. *The elementary functions* $\sin nz$, $\cos nz$, *and* $e^{\pm inz}$ *are the building blocks. Any linear combination of the above. Nonlinear functions too, for example*

$$\frac{1}{1+\sin^2 z}$$

is analytic in any strip on which $\sin z \neq \pm i$. *An entire function* $h = \sum_0^\infty a_n\, z^n$ *yields the entire example*

$$h(e^{iz}) \ = \ \sum_0^\infty a_n\, e^{inz}\,.$$

The last example can be modified to yield the general case as follows. Consider the mapping

$$w \ = \ e^{iz}\,. \tag{3}$$

It maps the strip (2) in the complex z plane to the annulus

$$\{w \ : \ e^{-a} < |w| < e^{a}\} \tag{4}$$

in the w plane. It maps the real axis in the z plane infinitely often around the unit circle in the w plane, the preimages of a point $w = e^{i\theta}$ are the points $z = \theta + 2\pi n$ with $n \in \mathbf{Z}$. Since the derivative dw/dz is nowhere zero it follows that the mapping is locally invertible with analytic inverse. The local inverses are branches of the function $z = (\ln w)/i$.

Theorem. *The correspondence*

$$f(z) \ = \ g(e^{iz}) \tag{5}$$

establishes a one to one correspondence between the 2π *periodic analytic functions* $f(z)$ *in the strip (2) and the analytic functions* $g(w)$ *on the annulus (4).*

Proof. That each such g yields an analytic periodic f on the strip and that distinct functions g yield distinct f is clear. What needs to be shown is that every periodic analytic function on the strip has such a representation.

Suppose that f is analytic and periodic in the strip (2). For each point w in the annulus (4) the preimages z under the map (3) lie in the strip and differ by integer multiples of 2π. Thus, the function f has the same value at all the preimages. It follows that a function g on the annulus is well defined by the formula $g(w) = f(z)$ since it does not matter which z one takes.

For any $\underline{w}$ choose a preimage $\underline{z}$. The Inverse Function Theorem implies that w has a local inverse $z = F(w)$ analytic on a neighborhood of $\underline{w}$ and satisfying $F(\underline{w}) = \underline{z}$. Near $\underline{w}$, $g(w) = f(F(w))$ is therefore analytic. This shows that g is analytic at every $\underline{w}$ so that g provides the desired representation of f. ∎

Theorem. *If $f(z)$ is a 2π periodic analytic function in the strip (2) then f has a Fourier series representation*

$$f(z) \;=\; \sum_{n=-\infty}^{\infty} c_n\, e^{inz}\,, \tag{6}$$

with coefficients given by the formulas

$$c_n \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, e^{-in\theta}\, d\theta\,. \tag{8}$$

Proof. Choose g so that (5) holds. Then use the Laurent expansion of g

$$g(w) \;=\; \sum_{-\infty}^{\infty} c_n\, w^n\,, \qquad c_n \;=\; \frac{1}{2\pi i}\oint_{|w|=1} \frac{g(w)}{w^{n+1}}\, dw\,. \tag{9}$$

Since $f(z) = g(e^{iz})$, one has

$$f(z) \;=\; \sum_{-\infty}^{\infty} c_n\, (e^{iz})^n$$

which is formula (6).

Parameterizing the curve $|w| = 1$ by $w = e^{i\theta}$ with $0 \le \theta \le 2\pi$, one has $dw = i\, w\, d\theta$ and the formula for c_n becomes

$$c_n \;=\; \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(e^{i\theta})}{w^{n+1}}\; i\, w\, d\theta \;=\; \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)}{w^n}\; d\theta \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, e^{-in\theta}\; d\theta, \,,$$

proving (8). ∎

This is not the way Fourier series were discovered. However, it does show that there is a deep connection between Fourier series and the theory of analytic functions.

§2. Fourier series of smooth periodic functions.

To derive the Fourier representation of smooth periodic and L^2 periodic functions the next result suffices. The key step uses the Fourier expansion of an approximating analytic periodic function.

Theorem. *If $f(\theta)$ is a smooth 2π periodic function whose Fourier coefficients all vanish, then $f = 0$.*

Proof. Choose normalizing coefficients C_ϵ so that $C_\epsilon e^{-x^2/\epsilon^2} \;\to\; \delta$ as $\epsilon \to 0$. Define

$$f_\epsilon(z) \;:=\; C_\epsilon \int_{-\infty}^{\infty} e^{-(z-\theta)^2/\epsilon^2}\; f(\theta)\; d\theta\,.$$

Then f is entire analytic in z and 2π periodic.

The Fourier coefficients of f_ϵ vanish since,

$$\int_0^{2\pi} e^{-in\theta}\; f_\epsilon(\theta)\; d\theta \;=\; C_\epsilon \int_0^{2\pi} e^{-in\phi}\; e^{-(\phi-\theta)^2/\epsilon^2}\; f(\theta)\; d\theta\; d\phi\,.$$

Change variable from θ to $\xi = \phi - \theta$ to show that the integral is equal to,

$$\int_0^{2\pi} e^{-in\phi}\; e^{-\xi^2/\epsilon^2}\; f(\xi+\phi)\; d\phi\; d\xi\,.$$

The change of variable $\eta = \xi + \phi$ shows that the $d\phi$ integral is equal to,

$$\int_0^{2\pi} e^{-in\phi}\; f(\xi+\phi)\; d\phi \;=\; \int_\xi^{\xi+2\pi} e^{-in(\eta-\xi)}\; f(\eta)\, d\eta \;=\; e^{in\xi}\int_\xi^{\xi+2\pi} e^{-in\xi}\; f(\xi)\; d\xi \;=\; 0,$$

since the Fourier coefficients of f vanish.

The Laurent expansion of analytic periodic functions then implies that $f_\epsilon = 0$.

On the other hand, as $\epsilon \to 0$, the restriction of f_ϵ to the real axis converges uniformly to f proving that $f = 0$. ∎