The Dog on a Leash Principal

Suppose that

$$m(t) \ : \ [0,1] \to \mathbf{R}^2 \setminus \{(0,0)\}$$

is a continuous nonvanishing function with $m(0) = m(t)$. The function $m(t)$ describes the closed path of a man that winds exactly n times around a flagpole at the origin.

Suppose at the same time the man's dog also describes a closed continuous path $d(t)$ for $0 \le t \le 1$.

Finally, suppose that the dog is on a leash which is always kept shorter than the distance to the flagpole, that is

$$\text{dist}(m(t), d(t)) \ < \ \text{dist}\big(m(t), (0,0)\big)\,.$$

I hope that with this description, the following result is geometrically obvious.

Theorem. *Under these hypotheses the dog also walks around the flagpole n times. That is, the closed path taken by the dog winds exactly n times around the origin.*

Rouche's Theorem in §65 is the special case

$$m(t) = f(\gamma(t)) \quad \text{and} \quad d(t) = f(\gamma(t)) + g(\gamma(t))$$

where $\gamma : [0,1] \to \mathbf{R}^2 \setminus \{(0,0)\}$ is a parametrization of the curve C.

Discussion. That the winding numbers are equal does not require any analyticity. That the winding numbers are exactly equal to the number of zeroes counting multiplicity is a consequence of the argument principal which does require analyticity.