Laurent Series Yield Partial Fractions.

Summary. Partial fraction expansions of rational functions are used in first year calculus and in complex analysis to find antiderivatives of rational functions and in ordinary differential equations when implementing the Laplace Transform methods. It permits one to compute the inverse Laplace Transform of any rational function. In this note we show that the fact that every rational function has a Partial Fraction Decompositions is a consequence of Laurent Expansions together with Liouville's Theorem.

A rational function is a function of the form

$$\frac{N(z)}{D(z)}$$

where both the numerator and the denominator are polynomials in z and D is monic, that is $D(z) = z^p +$ lower order terms. It follows that the function is analytic except possibly at the points z that are roots of the denominator. Denote those roots as

$$r_1, r_2, \ldots, r_n \tag{1}$$

with multiplicities

$$m_1, m_2, \ldots, m_n \tag{2}$$

so that

$$D = \Pi_{j=1}^n (z-\lambda)^{m_j}\,, \qquad m_1 + \ldots + m_n \;=\; \text{degree of } D\,.$$

Long division of the numerator N by the denominator D yields a quotient polynomial Q and a remainder polynomial R so that

$$\frac{N(z)}{D(z)} \;=\; Q(z) + \frac{R(z)}{D(z)}\,, \qquad \text{where} \qquad \deg Q = \deg N - \deg D\,, \qquad \text{and,} \qquad \deg R < \deg D\,.$$

Partial Fractions Decomposition Theorem. *Suppose that $R(z)/D(z)$ is a rational function with degree of R less than the degree of D. Denote by r_j the distinct roots of the denominator D and m_j their multiplicities. Let $S_j(z)$ denote the singular part of the Laurent expansion of R/D at the root r_j. The singular point r_j is either removable or a pole of order $\le m_j$ and*

$$\frac{R(z)}{D(z)} \;=\; \sum_j S_j(z)\,.$$

Proof. Define

$$f(z) \;:=\; \frac{R(z)}{D(z)} \;-\; \sum_j S_j(z)\,. \tag{3}$$

It suffices to show that $f = 0$.

The function $R(z)/D(z)$ is analytic at all points of the complex plane with the possible exception of the roots r_j.

Since the denominator has a zero of order m_j at r_j, R/D has a pole of order at most m_j at r_j so each singular part S_j is a finite sum. Therefore $\sum S_j$ is analytic on $\mathbf{C} \setminus \{r_1, \ldots, r_n\}$.

Since the degree of D is larger than the degree of R and the singular parts are combinations of negative powers of $z - r_j$, one has

$$\lim_{|z|\to\infty} |f(z)| \;=\; 0\,. \tag{4}$$

In particular, there is a radius $\rho > 0$ so that

$$|f(z)| \leq 1 \qquad \text{for} \qquad |z| \geq \rho\,. \tag{5}$$

In a neighborhood of r_j, Laurent's expansion shows that

$$\frac{R(z)}{D(z)} \;=\; S_j(z) + \text{convergent power series}\,.$$

The convergent power series is analytic in a neighborhood of r_j. For $k \neq j$ the functions $S_k(z)$ are also analytic in a disk centered at r_j. Thus

$$f(z) = \Big(\frac{R(z)}{D(z)} - S_j(z)\Big) - \sum_{k\neq j} S_k(z)$$

is the sum of terms each analytic on a disk centered at r_j so is itself analytic on the smallest of the finite set of disks. This proves that the function $f(z)$ has removable singularities at each of the r_j. Thus defining f appropriately at the r_j yields a function analytic on the entire complex plane.

In particular, f is continuous and therefore bounded on the disk $|z| \leq \rho$. Thus there is a constant $K > 0$ so that

$$|f(z)| \leq K \qquad \text{for} \qquad |z| \leq \rho\,. \tag{6}$$

Combining (5) and (6) shows that f is a bounded entire function. By Liouville's Theorem, $f(z)$ is a constant function. Then (4) shows that this constant value must be 0. Thus $f = 0$ completing the proof. ∎

Example. Find the partial fraction decompostion of $1/(z^2 + 1)$.

Solution. The denominator vanishes at $z = i$ and $z = -i$. Need the singular parts of the Laurent expansions at these points. One has

$$\frac{1}{z^2+1} \;=\; \frac{1}{(z+i)(z-i)} \;=\; \frac{1}{z-i}\,\frac{1}{z+i}\,.$$

To compute the Laurent series at $z = i$, which is a series in powers of $z - i$, need the Taylor series of

$$\frac{1}{z+i} \;=\; \frac{1}{(z-i)+2i} \;=\; \frac{1/2i}{1+\frac{z-i}{2i}}$$

about $z = i$. For $|z - i| < 2$, this is the sum of a geometric series

$$\frac{1/2i}{1+\frac{z-i}{2i}} \;=\; \frac{1}{2i}\Big(1 - \frac{z-i}{2i} + \Big(\frac{z-i}{2i}\Big)^2 \;\cdots\;\Big)\,.$$

Therefore,

$$\frac{1}{z^2+1} \;=\; \frac{1}{z-i}\;\frac{1}{2i}\Big(1 - \frac{z-i}{2i} + \Big(\frac{z-i}{2i}\Big)^2 \;\cdots\;\Big)\,,$$

so the singular part is equal to

$$\frac{1}{z-i}\,\frac{1}{2i}\,.$$

Here is a shortcut. You are looking at

$$\frac{1}{z-i}\,\frac{1}{z+i} \;=\; \frac{1}{z-i}\Big[a_0 + a_1(z-i) + \cdots\Big]\,.$$

All that you need is a_0. That coefficient is equal to the value of $1/(z+i)$ at $z=i$, hence $1/2i$.

Exercise. Use the short cut to compute the singular part at $z=-1$ and complete the partial fraction expansion.