1. The second exercise of §2 of the Dimension 1 handout.

2. 157/9.

3. (Variant of 156/5, 405/6). What does the fundamental existence and uniqueness theorem guarantee about the initial value problem $x' = x/t, \ x(1) = a$ with $a \in \mathbb{R}$. **Hint.** The equation is linear.

4-5. This problem concerns the logistic equation $x' = ax(1-x)$ with $a > 0$.

   a. Find the linearized equation\(^1\) at the equilibrium $x = 0$. Explain why this suggests instability of the equilibrium.

   b. Find the linearized equation at the equilibrium $x = 1$. Explain why this suggests stability of the equilibrium.

   The solution of the initial value problem (see text page 5)

   $$x' = ax(1-x), \ x(0) = A \quad \text{is} \quad x(t) = \frac{A e^{at}}{1-A + A e^{at}}.$$ 

   c. Find the linearized equation at this solution.

**Discovering Euler’s method.** One can seek solutions of systems that take values on a line through the origin. The next two problems are variants of problem 273/12 in M. Braun *Differential Equations and their Applications*. They show that for linear homogeneous autonomous systems this leads to Euler’s method.

6. Suppose that $\phi(t)$ is a complex valued differentiable function of time and $0 \neq u \in \mathbb{C}^N$. Prove that if $A$ is an $N \times N$ matrix and $X = \phi(t) u$ is the solution of the homogeneous linear system $X' = AX$ and $\phi(t) \neq 0$ for one time $t$, then $\phi(t) \neq 0$ for all $t$.

7. With the notation of the preceding problem, if $x = \phi(t) u$ is a solution of $X' = AX$ show that $\phi$ must be of the form $c e^{rt}$ for constants $c, r$.

8. **Free fall with small nonlinear friction.** A particle of mass $m$ falls freely under the force of gravity according to $mz'' = -mg$ where $g$ is the gravitational constant. Consider the effects of adding a small nonlinear frictional resistance. The frictional force opposes the motion,

   $$mz'' = -mg - \varepsilon |z'|, \quad z(0) = h > 0, \quad z'(0) = 0, \quad 0 < \varepsilon << 1.$$ 

   Compute the leading order correction. That is compute an approximate solution whose leading term is the unperturbed solution and the correction term is the first corrector in perturbation theory that is not identically equal to zero.

---

\(^1\) The linearization or variational equation of $X' = F(t, X)$ at a solution $\underline{X}(t)$ is the equation $U' = A(t) U$ with $A(t) = DX F(t, \underline{X}(t))$. 

Discussion. This is an example of a practical problem for which the nonlinear function is only differentiable a finite number of times.