1. In class we used that if $k \geq 0$ and $\alpha > 0$ then $g(t) := t^k e^{-\alpha t}$ is bounded on $t \geq 0$. Compute its maximum by finding that there is a $t_0$ so that $g$ increases to the left of $t_0$ and decreases to the right.

2. a. Draw the phase line for

$$x' = -x(x^2 - \varepsilon^2), \quad 0 < \varepsilon << 1.$$ 

Explain why observers not sensitive to small scales might be deceived into thinking that the equilibrium at $0$ is stable while it is not.

b. In contrast explain that the equilibrium $x = 0$ for $x' = x(x^2 - \varepsilon^2)$ is stable but to all but the most careful experimenter, appears unstable.

3. For the two spring system of 136/7 show that if the mass whose location is $x_1$ is subject to friction, that is it has an extra force $-ax'_1$ with $a > 0$, then all the eigenvalues of the corresponding $4 \times 4$ matrix have strictly negative real part.

Hint. You can do this by brute force. Or you can reason as follows.

- Introduce a natural energy and derive the dissipation law for solutions,

$$\frac{dE}{dt} = -a |x'_1|^2. \tag{1}$$

The kinetic energy is clear. The potential energy comes from the expansion or compression of the springs. Add the kinetic and potential energies. It should be conserved for $a = 0$. Equation (1) implies that the energy of all solutions is decreasing.

- Use decrease of energy to prove that all eigenvalues lie in $\{ \text{Re } z \leq 0 \}$.

- If there were a purely imaginary eigenvalue there would be a nonzero solution $U(t) := e^{i \omega t} v$. Show that the real and imaginary parts of $U(t)$ are solutions whose energy is periodic. Show that the energy of those solutions must be constant.

- Use the dissipation identity to show that a solution with constant energy has constant $x_1$. Show that a solution with constant $x_1$ is identically equal to zero.

Discussion. This is a non trivial and physically reasonable example of asymptotic stability.

4-5. 184/1

6. The second exercise of §5 of the Dimension 1.5 posting.

7. 185/5.