1. 135/1h.

2. 184/1i,ii,iii. **Hint.** iii is exactly solvable. i use phase lines. The example on pg 162 is a model for ii.

3. 137/9.

4. Prove the **Cayley-Hamilton Theorem.** If \( A \) is an \( N \times N \) matrix and \( p(z) = \det(zI - A) \) is its characteristic polynomial, then \( p(A) = 0 \).

**Remarks.** i. Since \( A^j \) is an \( N \times N \) matrix, \( p(A) \) is a sum of such matrices. ii. In class the two by two case was proved by approximating by matrices with distinct eigenvalues then passing to the limit. That also works in the \( N \times N \) case. iii. The suggestion of the next hint is a better proof relying on generalized eigenspaces. **Hint.** Factor

\[ p(z) = \prod_{j=1}^{k} (z - \lambda_j)^{m_j} \]

with distinct \( \lambda_j \). Then

\[ p(A) = \prod_{j=1}^{k} (A - \lambda_j I)^{m_j} . \]

Use the Spectral Decomposition Theorem to show that \( p(A)v = 0 \) for each vector \( v \) in a generalized eigenspace \( X_\mu \). Then represent an arbitrary vector as a sum of these.

5. 136/7 (Variant). I will modify part (d) and add a part (e).

(d) With \( \omega_1, \omega_2 \) as in the text explain why all orbits are periodic when \( \omega_1/\omega_2 \) is rational.

(e) Suppose now that \( \omega_1/\omega_2 \) is irrational. If you plot a typical orbit in the four dimensional phase space for a long time, what do you expect to find? If you plot in a three dimensional section (much easier to do) say in the \( x_1, x_2, \dot{x}_1 \) three space what do you expect to see? Finally explain why the plot of a two dimensional projection, say on the \( x_1, x_2 \) plane, will resemble the figures on the top of page 118 of HSD.

**Discussion.** This question should help to resolve the uneasiness of some students with studying the uncoupled oscillators of §6.2. In the small oscillations approximation of mechanical systems in a neighborhood of a strict minimum of potential energy (we will discuss this later), all systems look like uncoupled oscillators after clever coordinate change.

6. 137/11.

7. 186/7. At \( a = -1 \) means for \( a \) on a neighborhood of \( a = -1 \).