Instructions.
1. Clearly explain your answers.
2. You are allowed two sides of a 3” × 5” card of notes.
3. No calculators.
4. There are four problems.
Good Luck.

1. The first three parts concern the scalar ordinary differential equation
   \[ p' = p^{100} - 1. \]
   
   i. (4 points) Find all equilibrium points.
   ii. (4 points) Draw the phase line diagram.
   iii. (2 points) Determine the stability of each equilibrium point.
   iv. (3+2 points) This equation is in the family of equations
       \[ p' = p^{100} + a, \quad -\infty < a < \infty. \]

   Determine all values of \( a \) where bifurcations occur and describe the change(s) that occur at the bifurcations. Diagrams can help here.

Solution outline. The equilibria are the solutions of \( p^{100} - 1 = 0 \). There are two, \( p = \pm 1 \).
Since \( p^{100} - 1 \) is positive outside \([-1, 1]\) and negative in \([-1, 1]\), solutions move to the right when they are outside \([-1, 1]\) and move to the left in the interval \([-1, 1]\).
It follows that the equilibrium \( p = -1 \) is a sink and \( p = 1 \) is a source. This can also be shown by linearization since the equilibria are hyperbolic.
The same analysis works for the family in iv. when \( a < 0 \) and the equilibria are at \( \pm |a|^{1/2} \).
When \( a = 0 \) there is exactly one equilibrium, at \( p = 0 \). The phase line in this case is moving to the right at all points other than \( p = 0 \).
For \( a > 0 \) there are no equilibria and the flow is strictly right moving at all points.
2. This problem has four parts, ai, aii, bi, bii each worth 4 points. For each of the two systems

\[(a) \quad X' = \begin{bmatrix} -1 & 2 \\ -1/2 & -1 \end{bmatrix} X, \quad (b) \quad X' = \begin{bmatrix} 4 & 5 \\ -2 & -3 \end{bmatrix} X,\]

i. Determine if the system falls into one of the following categories, spiral sink, spiral source, saddle, nonspiral source, nonspiral sink.

ii. If a spiral sink or spiral source determine the direction of rotation of the spiral and the exponential rate of growth or decay.

If a source determine the direction of fastest growth. If a (nonspiral) sink determine the direction of slowest decay.

If a saddle determine the direction of the stable line. (Definition. The *stable line* is the set of initial data with the property that the solution of the corresponding initial value problem tends to 0 as \(t \to \infty\).)

**Solution outline.** a. Compute

\[\det(A - \lambda I) = \lambda^2 + 2\lambda + 2.\]

The eigenvalues are \(\lambda = -1 \pm i\). The equilibrium is a spiral sink.

The tangent at \((1, 0)\) has second component equal to \(-1/2\) so the orbit crosses the \(x\)-axis moving in the clockwise direction.

The solutions decay like \(e^{-t}\) thanks to the real part of the eigenvalues.

b. 

\[\det(A - \lambda I) = \lambda^2 + \lambda - 2 = (\lambda - 2)(\lambda + 1).\]

The eigenvalues are \(-1, 2\). The equilibrium is a saddle.

The stable line is the line of eigenvectors with eigenvalue \(-1\). That is, \(\mathbb{R}(1, -1)\).
3. (15 points) Define

\[
A = \begin{bmatrix}
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & -1 & 1
\end{bmatrix}.
\]

Find the general solution of \(X' = AX\).

**Solution outline.** Since

\[
A = \begin{bmatrix}
A_{3\times3} & 0 \\
0 & A_{2\times2}
\end{bmatrix},
\]

\[
\det(A - \lambda I) = \det(A_{3\times3} - \lambda I_{3\times3}) \det(A_{2\times2} - \lambda I_{2\times2}) = (\lambda + 1)^3[(1 - \lambda)^2 + 4].
\]

The roots are \(\lambda = -1, -1, -1, 1 \pm 2i\).

The eigenvectors with eigenvalue \(1 + 2i\) are the nonzero multiples of \(V := (0, 0, 0, -2i, 1)\).

Two linearly independent solutions are

\[
e^{(1+2i)t} V, \quad e^{(1-2i)t} \nu,
\]

For \(\lambda = -1\) the eigenspace is

\[
E_\lambda = \ker(A + I)^3 = \ker(A + I)^2 = \left\{v : v_4 = v_5 = 0\right\}.
\]

This set is invariant and \((A + I)^2\) vanishes on these vectors so the solutions with values in \(E_\lambda\) are given by the three parameter family of solutions

\[
e^{-t} \left[I + (A + I)t\right](a, b, c, 0, 0).
\]

The general solution is then

\[
e^{-t} \left[I + (A + I)t\right](a, b, c, 0, 0) + d e^{(1+2i)t} V + f e^{(1-2i)t} \nu.
\]
4. (5 points) Compute the linearization of the nonlinear system

\[ \begin{align*}
  x'_1 &= \cos x_1 \sin x_2, \\
  x'_2 &= \pi x_1^2 - x_2,
\end{align*} \]

at the equilibrium point \((x_1, x_2) = (1, \pi)\).

**Solution outline.** Define

\[ \begin{align*}
  f_1(x_1, x_2) &= \cos x_1 \sin x_2, \\
  f_2(x_1, x_2) &= \pi x_1^2 - x_2.
\end{align*} \]

The linearization at an equilibrium is the system \(X' = AX\) with matrix \(A\) equal to the value of

\[ \begin{bmatrix}
  \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
  \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix}, \]

evaluated at the equilibrium point.

The matrix of partial derivative is equal to

\[ \begin{bmatrix}
  -\sin x_1 \sin x_2 & \cos x_1 \cos x_2 \\
  2\pi x_1 & -1
\end{bmatrix}. \]

The matrix \(A\) is computed by plugging in \((x_1, x_2) = (1, \pi)\).