

Causality

Abstract Ordinary differential equations arise as models in an astounding number and variety of applications. This is not an accident. Any smooth stable system with a finite number of degrees of freedom **must** be modelled by an ordinary differential equation. This handout shows why.

1 Definitions.

Definition 1.1. *A system is called **causal** if for any t_0 the state at time t_0 uniquely determines the state at time $t \geq t_0$. It is called **stable** or **continuous** if the state for $t_0 \leq t \leq t_1$ depends continuously in a suitable topology on its state on t_0 . It is called **smooth** if the state for $t_0 \leq t \leq t_1$ depends in (an infinitely) differentiable way on the initial state.*

This definition is quite vague about set the states belong to, what continuity or smoothness means, and what are appropriate t_1 . The qualitative idea is what is important.

Definition 1.2. *A system is said to have a **finite number of degrees of freedom** if its state is determined by a finite number, N , of real measurements.*

The values of those measurements yield a set of points in \mathbb{R}^N where N is the number of measurements. The states are in one to one correspondence with a subset of Euclidean space. The values of all other measurements are functions of the N determining measurements.

Definition 1.3. *If N is the smallest number possible, then the system is said to have N **degrees of freedom**.*

In this case we suppose that that subset is open. This means that slightly changing the values of the N parameters determining the system yields values corresponding for a slightly perturbed system. In this case the notions of continuity and smoothness can be taken to be that of Euclidean space.

Example 1.4. *For a Newtonian particle in \mathbb{R}^3 the state is determined by the six measurement corresponding to position and velocity, x, v . The seventh measurement, kinetic energy, K is a function of those. If one takes N equal to 7, with the measurements x, v, K as determining the state then the states correspond to six dimensional surface in \mathbb{R}^7 given by the equation*

$$K = m|v|^2/2.$$

The system has six degrees of freedom. For the six measurements the states correspond to all of \mathbb{R}^6 .

2 Connection with ODE

Suppose that a system has N degrees of freedom and its states correspond to the open set of points $\Omega \subset \mathbb{R}^N$. Suppose that it is smooth and causal in the sense that for each $x_0 \in \Omega$ and t_0 there is a $t_1 > t_0$ and an $r > 0$ so that for $|x - x_0| < r$ there is a unique future state $\phi((t, t_0, x_0))$ defined for $t_0 \leq t \leq t_1$. And that ϕ is an infinitely differentiable function of its arguments.

Define

$$F(t, x) = \left. \frac{\partial \phi}{\partial t} \right|_{(t, t_0, x_0)}.$$

F is the time derivative at time t of the trajectory which at time t is equal to x . The definition implies that every trajectory of the causal system satisfies,

$$x' = F(t, x)$$

showing that smooth causal systems are all described by a smooth system of ordinary differential equations.

A smooth causal system with N degrees of freedom is always modeled by a system of N first order ordinary differential equations.